Lecture Notes on General Relativity

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Preface and recommended literature

These lecture notes have been prepared for a new course on general relativity with applications to relativistic astrophysics in the Guelph–Waterloo Institute for Physics (GWIP), the joint graduate school of the Universities of Guelph and Waterloo, as well as Perimeter Institute for Theoretical Physics (Fall 2019). This course assumes familiarity with special relativity and associated mathematical methods, as, e.g., described in Chapters 1–4 of

- B. F. Schutz, *A first Course in General Relativity* (Cambridge, 2012)

The recommended primary textbook references for this graduate-level course are the following:

- N. Straumann, *General Relativity* (Springer, 2013)
- S. M. Carroll, *Spacetime and Geometry: An introduction to General Relativity* (Pearson, 2018)

While the former is more mathematical and concise in tone, the latter is much more descriptive and chooses a more intuitive point of view. A combination of both books should hopefully provide a good foundation, in addition to these lecture notes.

Many textbooks helped me compose these lecture notes. In addition to the above mentioned, I point out the following two (in German only):

- R. Oloff, *Geometrie der Raumzeit* (Vieweg, 2008)
- T. Fliessbach, *Allgemeine Relativitätstheorie* (Elsevier, 2006)

Further recommended textbooks include

- E. Poisson, *Gravity* (Cambridge, 2014)
- R. Wald, *General Relativity* (U Chicago Press, 1984)
- C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation* (Freeman, 1973)

Finally, for the mathematical sections I recommend the following introductions to differential geometry and Riemannian geometry:

- J. M. Lee, *Introduction to Smooth Manifolds* (Springer, 2014)
- J. M. Lee, *Riemannian Manifolds: An Introduction to Curvature* (Springer, 1997)
- S. Gallot, D. Hulin, J. Lafontaine, *Riemannian Geometry* (Springer 2004)

Chapter 1

Prelude

1.1 Newton's Theory of Gravitation

In 1687, Newton published his most famous, three-volume work "Philosophiae naturalis principia mathematica", one of the most influential books in science of all times. Newton states his three laws of motion, setting the foundation for classical mechanics. He derives his law of universal gravitation, thereby explaining Galilei's laws of free fall and Kepler's laws of planetary motion in one unified theory. Furthermore, he also introduced his concepts of absolute space, absolute time, and action at a distance, which have been very influential until Einstein's days (more on this later).

Newton's law of universal gravitation states that the motion of N point masses m_i (any number of objects in the Universe) is given by

$$m_i \frac{\mathrm{d}^2 \mathbf{r}_i}{\mathrm{d}t^2} = -G \sum_{j=1, j\neq i}^{N} \frac{m_i m_j (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3}. \tag{1.1}$$

Here, $\mathbf{r}_i(t)$ refers to the position of particle i at time t. The gravitational constant G is experimentally determined to[1]

$$G = 6.67430(15) \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}. \tag{1.2}$$

Note the following properties of the gravitational force introduced on the right hand side of Eq. (1.1):

- The force is attractive and acts radially along the separation of two bodies $\mathbf{r}_i - \mathbf{r}_j$.
- The force is proportional to the product of the two masses.
- The force is long-range and is proportional to the distance squared.
- With the numerical value of the proportionality constant (1.2), the gravitational force is by far the weakest of the four fundamental interactions. This can be illustrated, for instance, by comparing the gravitational and electrostatic force between two protons:

$$\frac{G m_\mathrm{p}^2}{r^2} = \alpha^{-1} \frac{m_\mathrm{p}^2}{M_\mathrm{Pl}^2} \frac{e^2}{r^2} = 0.8 \times 10^{-36} \frac{e^2}{r^2}. \tag{1.3}$$

[1] 2018 CODATA recommended values, https://physics.nist.gov/Constants

Here, $\alpha = e^2/\hbar c \simeq 1/137$ is the fine structure constant. In other words, the weakness of the gravitational force is due to the Planck mass

$$M_{\rm Pl} = \left(\frac{\hbar c}{G}\right)^{1/2} = 1.2\times 10^{19}\frac{\rm GeV}{c^2} \tag{1.4}$$

being huge compared to mass scales of particle physics (regarding the above example, $m_{\rm p} = 0.938\,{\rm GeV}/c^2$).

One can reformulate Eq. (1.1) as an equation of motion in a **gravitational field**,

$$m\frac{{\rm d}^2\mathbf{r}}{{\rm d}t^2} = -m\nabla\Phi(\mathbf{r}), \tag{1.5}$$

where $\Phi(\mathbf{r})$ is the **gravitational potential**,

$$\Phi(\mathbf{r}) = -G\sum_j^N \frac{m_j}{|\mathbf{r}-\mathbf{r}_j|} = -G\int {\rm d}r'^3\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.6}$$

Here, we have taken the continuum limit to a mass density ${\rm dm_j} \to \rho(\mathbf{r}'){\rm dr}'^3$. Equation (1.6) implies the gravitational **field equation**

$$\Delta\Phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r}), \tag{1.7}$$

a linear partial differential equation of second order for the gravitational potential. Note that the source of the gravitational potential is the mass density on the right hand side.

Exercise 1.1.1. *Derive the field equation* (1.7)*.*

It is obvious that Newton's theory of gravitation cannot be strictly valid, as there are contradictions with special relativity. The force postulated on the right hand side of Eq. (1.1) is acting instantaneously at a distance, i.e., there is instantaneous communication. This is in contrast to the third fundamental postulate of special relativity, namely that the speed of light be constant in all inertial frames; there is no faster-than-light communication between inertial observers. As a result, Newton's gravitational force $F = GMm/r^2$ is not invariant under Lorentz transformations, i.e., when transforming M, r, F to another inertial frame, $F' \neq GM'm'/r'^2$.Therefore, Newton's theory of gravity can only hold in the non-relativistic limit of a more general theory of gravitation—the theory of general relativity developed by Albert Einstein between 1907 and 1915.

1.2 The equivalence principle and the road to General Relativity

Let us start by pointing out a notable analogy between Newton's theory of gravitation and the equation of motion and the field equation of electrostatics,

$$m\frac{{\rm d}^2\mathbf{r}_i}{{\rm d}t^2} = -q\nabla\Phi_{\rm e}, \tag{1.8}$$
$$\Delta\Phi_{\rm e} = -4\pi\rho_{\rm e}. \tag{1.9}$$

Here, q denotes the charge of a particle of mass m, and $\Phi_{\rm e}$ is the electrostatic potential. In analogy to Eq. (1.7), the electric field is sourced by the charge density $\rho_{\rm e}$. Note that the particle charge, which appears as coupling constant in Eq. (1.8), is independent of the particle's mass. Likewise one could suspect that the **gravitational mass** m on the right-hand side of Eq. (1.5) is independent of the **inertial mass** m on the left-hand side, i.e., that they are distinct independent properties of a body. This would obviously be consistent with Newton's theory, as the theory itself does not put any constrains on the nature of these coupling constants. However, already Galileo had realized that all bodies fall at the same rate, i.e., that the inertial and gravitational mass are proportional to each other. This means that these masses are 'equivalent'—they are equal for an appropriate choice of units. Newton himself has verified this remarkable fact to a precision better than a part in one thousand by using pendulums loaded with test masses of different material (gold, silver, lead, glass), observing that the pendulum's period is indeed independent of the material.

Exercise 1.2.1. *Evaluate Newton's law* (1.5) *in the context of Galilei's free fall experiment. Show that Galilei's observation that all bodies fall at the same rate implies proportionality of the inertial and gravitational mass,* $m_{\text{grav}} \propto m_{\text{inert}}$.

Exercise 1.2.2. *Evaluate Newton's law* (1.5) *in the context of Newton's pendulum experiment. Show that for small displacements the period* T *is given by* $(T/2\pi)^2 = (m_{\text{inert}}/m_{\text{grav}})(l/g)$, *where* g *is the local gravitational acceleration and* l *the length of the pendulum.*

Around Einstein's time, Eötvös experimented with a torsion pendulum, originally built in 1890, with which he was able to obtain an accuracy of 5×10^{-9} by 1922.[2] Meanwhile, the equivalence of inertial and gravitational mass has been experimentally verified to an accuracy of of the order of 10^{-15} Touboul et al. (2017) (see Will (2006) for a review), which suggests the

Weak Equivalence Principle (WEP). *The motion of a test body in a gravitational field is independent of its mass and composition (neglecting effects of spin and higher mass moments).*

Einstein was struck by this 'coincidence' of equivalence in Newton's theory:

> "This law [...] now struck me in its deep significance. I wondered to the highest degree about its validity and supposed it to be the key to a deeper understanding of inertia and gravitation."[3]

Indeed, it turned out that Einstein would generalize the WEP and make the following, stronger version of the WEP to the foundational pillar of his general theory of relativity:

Einstein's Equivalence Principle (EEP). *In an arbitrary gravitational field local non-gravitational experiments cannot distinguish a freely falling nonrotating system (a local inertial system) from a uniformly moving system in the absence of a gravitational field.*

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[2] Einstein did not seem to have been aware of Eötvös' early results when he developed his theory: "I did not seriously doubt its [the law of equality of inertial and gravitational mass] strict validity even without knowing the result of the beautiful experiment of Eötvös" (A. Einstein in a lecture *On the Origins of the General Theory of Relativity*).

[3] A. Einstein in a lecture *On the Origins of the General Theory of Relativity.*

Example (elevator experiment): consider a person in a freely falling elevator. The equation of motion in the Lab frame (surrounding building, inertial system) is given by

$$m_{\rm inert}\frac{{\rm d}^2\mathbf{r}}{{\rm d}t^2} = m_{\rm grav}\mathbf{g}, \tag{1.10}$$

where $\mathbf{g}$ is gravitational acceleration. Transforming this into the co-moving frame of the elevator, $\mathbf{r} \to \mathbf{r}' = \mathbf{r} - (1/2)\mathbf{g}t'^2$, $t \to t' = t$, we obtain

$$m_{\rm inert}\frac{{\rm d}^2\mathbf{r}'}{{\rm d}t^2} = (m_{\rm grav} - m_{\rm inert})\mathbf{g}. \tag{1.11}$$

Note that the person in the elevator cannot distinguish between whether the elevator is accelerated by some external force (e.g., pulled by a rope) or whether the elevator is accelerated by the gravitational field. Equivalence of gravitational and inertial mass (forces) thus means

$$\frac{{\rm d}^2\mathbf{r}'}{{\rm d}t^2} = 0, \tag{1.12}$$

i.e., that gravitational forces can be eliminated.

Some remarks regarding the EEP:

- Note that this postulate generalizes the WEP from purely mechanical to *all* physical processes and to inhomogeneous gravitational fields. In a freely falling frame all processes occur as if there was no gravitational field. In other words, in a local inertial frame the laws of special relativity apply.

- The cancellation of acceleration and gravitational forces applies only to the center of mass of the freely falling frame, hence the restriction to 'local experiments'. In other words, gravity can be locally 'transformed away'. As we will see later, this gives rise to the **principle of minimal coupling**, through which general-relativistic laws can be obtained from their special-relativistic version. This principle allowed Einstein to derive general-relativistic predictions long before he had actually finished the theory.

- The EEP links inertia and gravitation such that they cannot be distinguished. Because of the equivalence of mass and energy, all forms of energy contribute to the inertial and gravitational mass.

- There is an even stronger version of the equivalence principle, the so-called **strong equivalence principle**. In addition to EEP, it also includes self-gravitating bodies and experiments involving gravitational forces, which we shall not discuss here.

Equipped with this postulate of equivalence as a crucial insight, how does one go about formulating a theory that is consistent with special relativity and that recovers Newton's laws of gravitation as a (still to be defined) limit? A first attempt may be to find a theory of gravity within the framework of special relativity. This is what Einstein and others attempted to achieve shortly after the 1905 publication of special relativity.

Some insight can again be gained from a comparison with the generalization of electrostatics (Eq. (1.9)) to the relativistic theory of electrodynamics. In order to avoid action at a distance in the dynamic theory, i.e., a change in the charge density $\rho_e(\mathbf{r}, t)$ results in an immediate change in the field $\Phi_e(\mathbf{r}, t)$ at arbitrary distance, one needs to make the replacement

$$\Delta \quad \rightarrow \quad \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \tag{1.13}$$

in Eq. (1.9), so that changes propagate with the speed of light c. The requirement to transformation between inertial systems with relative motion suggests that the charge density be generalized to a current density (as seen from an inertial observer not at rest with the charge distribution), the 4-current j^μ. We thus expect the following replacement in the source term of Eq. (1.9):

$$\rho_e \quad \rightarrow \quad j^\mu = (\rho_e c, \rho_e \mathbf{v}). \tag{1.14}$$

Here, $\mathbf{v}$ denotes the velocity field. The electric potential will therefore also need to be generalized to a 4-potential,

$$\Phi_e \quad \rightarrow \quad A^\mu = (\Phi_e, A^i), \tag{1.15}$$

where A^i denotes the components of the magnetic vector potential $\mathbf{A}$. One would then guess that the relativistic generalization of the field equation of electrostatics is given by

$$\Delta\Phi_e = -4\pi\rho_e \quad \rightarrow \quad \Box A^\mu = \frac{4\pi}{c} j^\mu. \tag{1.16}$$

Equation (1.16) is indeed equivalent to Maxwell's equations (provided gauge conditions for the potentials), and note that the 0-component indeed reduces to Eq. (1.9) in the static limit.

Exploiting the aforementioned analogy between Newton's theory of gravitation and electrostatics, one may attempt to proceed in the same way in order to generalize Newton's theory to a special relativistic theory of gravity. First, one would adopt the same replacement (1.13) in Eq. (1.9). In contrast to the charge of a particle, which is a Lorentz scalar (invariant under Lorentz transformations), the rest mass of a particle is not. While the charge density transforms as the 0-component of a Lorentz vector (note that $\rho_e = \Delta q/\Delta V$ obtains a factor γ due to length contraction during a Lorentz transformation), the rest mass density ρ transforms as the 00-component of a Lorentz tensor $T^{\mu\nu}$, which we shall call the energy-momentum tensor for now. Therefore, in analogy to Eq. (1.14), the appropriate substitution on the right-hand side of Eq. (1.9) would be

$$\rho \quad \rightarrow \quad T^{\mu\nu} \sim \begin{pmatrix} \rho c^2 & \rho c v^i \\ \rho c v^i & \rho v^i v^j \end{pmatrix}. \tag{1.17}$$

Accordingly, the generalization of the gravitational potential then has to lead to a tensor $g^{\mu\nu}$, which we shall call the metric tensor,

$$\Phi \quad \rightarrow \quad g^{\mu\nu}. \tag{1.18}$$

This would finally result in the following generalized gravitational field equations:

$$\Delta\Phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r}) \quad \rightarrow \quad \Box g^{\mu\nu} \sim G T^{\mu\nu}. \tag{1.19}$$

A proportionality constant on the right-hand side would need to be introduced and adjusted such that the 00-component in the static case reduces to Newton's field equation.

While we will later indeed find similar field equations in the limit of weak gravitational fields (the so-called **linearized field equations**), it is obvious that Eq. (1.19) cannot represent a fully generalized theory of gravitation:

- The equivalence of mass and energy in special relativity leads to additional complication not present in electrodynamics. The gravitational field itself carries energy which should correspond to a mass that sources the field. This would mean that the field equations should be intrinsically non-linear; Eq. (1.19) however, is purely linear in the generalized gravitational potentials.

- In addition, generalizing Newton's equation of motion (1.5) to special relativity is not straightforward either. Because of the mass-energy equivalence the inertial mass of a body may depend on the gravitational field itself, which means that also the equations of motion should become non-linear.

- The fact that inertia and gravitation cannot be distinguished due to the EEP points to the fact that any generalized theory should be a theory including non-inertial frames. It seems that non-inertial frames would be at the heart of any generalized theory. Special relativity, however, is a theory that is intricately linked to inertial frames, i.e., all laws of physics apply to inertial frames only and they are only invariant under Lorentz transformations between inertial frames.

Einstein first pursued a similar way as outlined above in trying to find a generalized version of Newton's field equations in special relativity, but then concluded similarly to the second point above:[4]

> "Like most authors at that time, I tried to formulate a field law for gravity, since the introduction of action at a distance was no longer possible [...]. The simplest and most natural procedure was to retain the scalar Laplacian gravitational potential and to add a time derivative to the Poisson equation in such a way that the requirements of the special theory would be satisfied. In addition, the law of motion for a point mass in a gravitational field had to be adjusted to the requirements of special relativity. Just how to do this was not so clear, since the inertial mass of a body might depend on the gravitational potential."

Einstein also showed that the equations of motion resulting from such a generalization would be in violation of the EEP. He concluded:[5]

> "I now gave up my previously described attempt to treat gravitation in the framework of the special theory as inadequate. It obviously did not do justice to precisely the most fundamental property of gravitation."[6]

We conclude that in order to construct a generalized theory of gravitation, one must go beyond special relativity, and find an entirely new framework, which

- respects the EEP and puts it as a corner stone of the theory

- leads to non-linear field equations for the generalized gravitational potentials

- holds in non-inertial frames, as indicated by the EEP, i.e., it should be possible to formulate the fundamental laws of the theory in any coordinate frame (**'general covariance'**)

[4] A. Einstein in a lecture *On the Origins of the General Theory of Relativity*.
[5] A. Einstein in a lecture *On the Origins of the General Theory of Relativity*.
[6] By 'most fundamental property' he meant the EEP.

One other important hint comes from the following observation: According to the WEP, all bodies experience the same gravitational acceleration $g = GM/r^2$ regardless of their own mass m. One may thus think of g as a property of space, rather than as of a gravitational force. What is then the fundamental property of space that is observed as gravitational acceleration, and how can such a property by defined in any given frame? How is this covariant property related to the source of gravity?

These conclusions and questions lead to the concept of manifolds and curvature; we shall discuss these mathematical concepts in the next section. Einstein himself had to learn these concepts eventually as he was trying harder to find a generalized theory of gravitation.

		1911		1912		1915
1907 EEP	$\rightarrow$	spacetime	$\rightarrow$	tensor analysis	$\rightarrow$	$R_{\mu\nu} = 0$ vacuum field eqs.
		curvature		in 4D spacetime		$G_{\mu\nu} = \kappa T_{\mu\nu}$ full field eqs.

Chapter 2

Mathematical Foundations

2.1 Manifolds

The concept of a manifold is a generalization of the familiar space $\mathbb{R}^n$. A manifold corresponds to a space that may be 'curved', but locally 'looks like' $\mathbb{R}^n$. The entire manifold is constructed by smoothly sewing together local regions that look like $\mathbb{R}^n$. This construction is based on charts that can be grouped into an atlas, which we shall now define.

Definition 2.1.1. *A **chart** or **coordinate system** of a (topological) space M is a bijective map $\phi : U \to U' \in \mathbb{R}^n$ that maps an open subset $U \subset M$ onto an open subset $U' \subset \mathbb{R}^n$. The coordinates $(x^1, \ldots, x^n)$ of the image $\phi(p) \in U'$ for any point $p \in U$ are called the **coordinates** of p in the chart. An indexed collection of charts $\mathcal{A} = \{(\phi_i, U_i)|i \in I\}$ is a $\mathcal{C}^k$ **atlas** if*

1. *the U_i cover M, that is, $\bigcup_{i \in I} U_i = M$*
2. *the charts are 'smoothly sewn together' or 'compatible', that is, if any charts overlap, $U_i \cap U_j \neq \emptyset$, the mapping $\phi_j \circ \phi_i^{-1}$: $\phi_i(U_i \cap U_j) \subset \mathbb{R}^n \to \phi_j(U_i \cap U_j) \subset \mathbb{R}^n$ must be bijective and $\mathcal{C}^k$, i.e., k-times continuously differentiable.*

Note that the map $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ in the above definition provides a relation between the two coordinates in $U_i \cap U_j$, i.e., where the charts overlap. It is thus called a **change of coordinates** or **coordinate transformation**. Furthermore, we note that since $\phi_{ij}^{-1} = \phi_{ji}$ on the mappings where they are defined, the inverses of all coordinate transformations are also smoothly differentiable, which means that they are **diffeomorphisms** (bijektive differentiable mappings whose inverse are also differentiable).

Definition 2.1.2. *Two $\mathcal{C}^k$-atlases $\mathcal{A}$ and $\mathcal{B}$ of a space M are called **equivalent**, if $\mathcal{A} \cup \mathcal{B}$ is a $\mathcal{C}^k$-atlas of M. A $\mathcal{C}^k$-atlas $\mathcal{A}$ is called **maximal**, if it contains every atlas that is equivalent to $\mathcal{A}$.*

Note that according Def. (2.1.1) this means that for two equivalent atlases, their charts are pairwise compatible, and that a maximal atlas contains all compatible charts. In particular, any atlas can be extended to a maximal atlas by adding all possible compatible charts.

Definition 2.1.3. *An n-**dimensional** $\mathcal{C}^k$-**manifold** $[M, \mathcal{A}]$ is a topological space M together with a maximal $\mathcal{C}^k$-atlas $\mathcal{A}$, whose charts (ϕ_i, U_i) map into $\mathbb{R}^n$. The number n is called the **dimension** of the manifold.*

Some remarks:

- The requirement of a maximal atlas in Def. (2.1.3) is necessary. This is because the same space equipped with two equivalent atlases would otherwise count as two different manifolds.
- The requirement of a topological space in the above definitions is necessary to ensure that open sets exist. The definition of a manifold, in principle, also requires that M be a Hausdorff space with a countable basis. These properties are needed to ensure uniqueness of limits (Hausdorff) and to allow for integration on manifolds (countable basis).
- For all practical purposes we will assume $k = \infty$. The degree of differentiability of a manifold is not crutial for our purposes.
- Since any atlas can be extended to a maximal atlas, it suffices to show that a given topological space admits any $\mathcal{C}^k$-atlas for it to be a $\mathcal{C}^k$-manifold.

Some simple examples of manifolds:

1. A trivial example for an n-dimensional $\mathcal{C}^\infty$-manifold is $\mathbb{R}^n$. The simplest maximal atlas is given by the single chart $(\mathbb{R}^n, \mathrm{Id})$, where Id is the identity map.
2. Consider the curve
$$M = \{(x, y) \in \mathbb{R}^2 : y = x^3\}. \tag{2.1}$$
The single chart (ϕ, M) with $\phi : M \to \mathbb{R}$, $\phi(x, y) = y$ is an atlas of M.

The following excercises explore a few more examples.

Exercise 2.1.4. *For the curve in the above example, consider another set of charts:*

$$U_1 = \{(x, y) \in M : y > 1\}, \ \ \phi_1(x, y) = y, \tag{2.2}$$
$$U_2 = \{(x, y) \in M : y < -1\}, \ \ \phi_2(x, y) = y, \tag{2.3}$$
$$U_3 = \{(x, y) \in M : -2 < x < 2\}, \ \ \phi_3(x, y) = x. \tag{2.4}$$

Show that these charts form an atlas. Furthermore, show that this atlas is not equivalent to the one defined in the above example.

Exercise 2.1.5. *Consider the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$. One can construct a chart from the open set U_1 defined to be the sphere minus the north pole via* ***stereographic projection****, i.e., by drawing a line from the north pole through any point $p \in U_1$ and assigning $\phi_1(p)$ the point of intersection of that line with the plane $z = -1$. Show that this mapping can be written as $\phi_1 : U_1 \subset S^2 \to \mathbb{R}^2$, with*

$$\phi_1(x, y, z) = \left(\frac{2x}{1 - z}, \frac{2y}{1 - z} \right). \tag{2.5}$$

Analogously, another chart (ϕ_2, U_2) can be defined by projection from the south pole to the plane $z = 1$. Check that

$$\phi_2(x, y, z) = \left(\frac{2x}{1 + z}, \frac{2y}{1 + z} \right). \tag{2.6}$$

Show that $\mathcal{A} = \{(\phi_i, U_i) | i = 1, 2\}$ provides an atlas for S^2.

2.2 Tangent space

This section introduces the concept of tangent spaces, a linear space that looks like $\mathbb{R}^n$ any manifold M can be endowed with at a given point $p \in M$. Our intuition leads us to think that such a space 'tangent' to a manifold in p would be the collection of all tangent vectors of any curve going through p. We shall start with the general, coordinate independent definitions of tangent vectors and the tangent space, and subsequently show how these abstract definitions relate to our intuitive notion of tangent vectors and spaces. In this section, M denotes an n-dimensional $\mathcal{C}^k$-manifold, where we will set $k = \infty$ for all practical purposes.

Definition 2.2.1. *(Space of all smooth functions on M) A real-valued function f on M, $f : M \to \mathbb{R}$, $p \in M \mapsto f(p) \in \mathbb{R}$, is called **smooth** ($\mathcal{C}^\infty$), if for any chart (ϕ, U) the function $f \circ \phi^{-1} : \phi(U) \subset \mathbb{R}^n \to \mathbb{R}$ is smooth. The linear space of all such smooth functions on M is denoted by $\mathcal{F}(M)$.*

Definition 2.2.2. *(Algebraic definition of the tangent space) Let $p \in M$. A function (operator) $x : \mathcal{F}(M) \to \mathbb{R}$ satisfying*

(i) $x(\lambda f + \mu g) = \lambda x(f) + \mu x(g)$

(ii) $x(fg) = (xf)g(p) + f(p)(xg)$

*for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in \mathcal{F}(M)$ is called a **tangent vector** at $p \in M$. The set of all tangent vectors at p is called the **tangent space** T_pM of M at p. The union of all local tangent spaces,*

$$TM \equiv \bigcup_{p \in M} T_pM \tag{2.7}$$

*is called the **tangent space** or **tangent bundle** of M.*

Theorem 2.2.3. *The tangent space T_pM is a linear (vector) space.*

Exercise 2.2.4. *Proof Theorem 2.2.3. Note that it suffices to show that any linear combination of tangent vectors is a tangent vector.*

Below we will show that tangent vectors at $p \in M$ can be identified with operators that provide the directional derivative along curves through p. The following theorem provides a property that is obvious in this context, namely that the (directional) derivative of a constant function vanishes.

Theorem 2.2.5. *Let $p \in M$, $x \in T_pM$ and $f \in \mathcal{F}(M)$ a constant function. Then $x(f) = 0$.*

Exercise 2.2.6. *Proof Theorem 2.2.5.*

In the following we shall develop a more intuitive characterization of the tangent space.

Definition 2.2.7. *A $\mathcal{C}^\infty$-**curve** c on M is a function $c : I \subset \mathbb{R} \to M$, such that $\phi \circ c : I \to \mathbb{R}^n$ is smooth for all charts ϕ.*

Every curve c with $c(0) = p$ gives rise to a tangent vector $x \in T_pM$,

$$x : \mathcal{F}(M) \to \mathbb{R} \tag{2.8}$$

$$f \mapsto (f \circ c)'(0). \tag{2.9}$$

Here, $'$ denotes the regular derivative for functions on $\mathbb{R}$. In order to check that x is indeed a tangent vector, we need to verify conditions (i) and (ii) in Def. 2.2.2 making use of the fact that the space of functions on $\mathbb{R}$ is a linear space and that they obey the product rule:

$$x(\lambda f + \mu g) = ((\lambda f + \mu g) \circ c)'(0) = \lambda (f \circ c)'(0) + \mu (g \circ c)'(0) = \lambda x(f) + \mu x(g), \tag{2.10}$$

$$\begin{aligned} x(fg) = ((fg) \circ c)'(0) &= ((f \circ c)(g \circ c))'(0) && (2.11)\\ &= (f \circ c)'(0)(g \circ c)(0) + (f \circ c)(0)(g \circ c)'(0) && (2.12)\\ &= x(f)g(p) + f(p)x(g). && (2.13) \end{aligned}$$

This observation motivates the following definition:

Definition 2.2.8. *Let $p \in M$, $x \in T_pM$ and $f \in \mathcal{F}(M)$, and $c : I \subset \mathbb{R} \to M$ a smooth curve on M with $c(0) = p$. We shall call $c'(0) = x$ the tangent vector generated by c at p as constructed above. The real number $x(f) = (f \circ c)'(0)$ is called **directional derivative** of f at p in direction of c.*

Making reference to a chart ϕ we can express the directional derivative $x(f)$ in direction of a curve c as a linear combination of certain tangent vectors:

$$x(f) = (f \circ c)'(0) = \big((f \circ \phi^{-1}) \circ (\phi \circ c)\big)'(0) = \sum_{i=1}^{n} \frac{\partial (f \circ \phi^{-1})}{\partial x^i}(\phi(p)) \frac{\mathrm{d}x^i}{\mathrm{d}t}(0), \tag{2.14}$$

where the x^i are defined by $\big(x^1(t), \ldots, x^n(t)\big) = \phi(c(t))$. This gives rise to the following definition:

Definition 2.2.9. *Given $p \in M$ and a chart ϕ of M containing p. We denote by $\frac{\partial}{\partial x^i}\Big|_p$ or $\partial_i|_p$ the tangent vector that maps any function $f \in \mathcal{F}(M)$ to the real number*

$$\begin{aligned} \partial_i|_p : \mathcal{F}(M) &\to \mathbb{R} && (2.15)\\ f &\mapsto \partial_i|_p(f) = \frac{\partial (f \circ \phi^{-1})}{\partial x^i}(\phi(p)). && (2.16) \end{aligned}$$

Applying the tangent vector $\partial_i|_p$ onto f thus means to compute the i-th partial derivative of the function $f \circ \phi^{-1}$. We will now show that $(\partial_1|_p, \ldots, \partial_n|_p)$ form a basis of T_pM. We start by deriving an identity that will prove useful in showing that $(\partial_1|_p, \ldots, \partial_n|_p)$ span T_pM.

Lemma 2.2.10. *For any smooth function on an open ball about the origin in $\mathbb{R}^n$, $g : B_0 \subset \mathbb{R}^n \to \mathbb{R}^n$, there exist smooth functions $g_1, \ldots, g_n$ such that*

$$g(x) = g(0) + \sum_{i=1}^{n} g_i(x) x^i, \quad g_i(0) = \frac{\partial g}{\partial x^i}(0). \tag{2.17}$$

Proof. This can be easily seen by writing

$$g(x) - g(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} g(tx^1, \dots, tx^n)\mathrm{d}t = \sum_{i=1}^n x^i \int_0^1 \partial_i g(tx^1, \dots, tx^n)\mathrm{d}t \tag{2.18}$$

$$\equiv \sum_{i=1}^n g_i(x) x^i, \tag{2.19}$$

where we have defined the functions g_i we were looking for in the last step. □

Theorem 2.2.11. *Let $p \in M$ and ϕ be a chart of M containing p. The tangent vectors $\partial_1|_p, \dots, \partial_n|_p$ form a basis of T_pM.*

Proof. (i) Linear independence: Let's consider the linear combination

$$\sum_{i=1}^n \lambda_i \partial_i|_p = 0, \tag{2.20}$$

i.e., for $f \in \mathcal{F}(M)$ we have

$$\sum_{i=1}^n \lambda_i \frac{\partial (f \circ \phi^{-1})}{\partial x^i}(\phi(p)) = 0. \tag{2.21}$$

Let us now choose the function f_j that maps every point p in the domain of ϕ onto its j-th coordinate in $\mathbb{R}^n$ in the chart ϕ, i.e., $f_j = \phi_j$, where $\phi : p \to (\phi_1(p), \dots, \phi_n(p))$. This yields $\lambda_j = 0$ for any $j = 1, \dots, n$.
(ii) $\partial_i|_p$ span T_pM: We now show that any tangent vector $x \in T_pM$ can be written as a linear combination of $\partial_i|_p$. As above, let f_j be the function that projects a point on its j-th coordinate in the chart ϕ, and let $\lambda_j = x(f_j)$. We will show that

$$x = \sum_{i=1}^n \lambda_i \partial_i|_p, \quad \text{i.e.,} \quad x(f) = \sum_{i=1}^n \lambda_i \frac{\partial (f \circ \phi^{-1})}{\partial x^i}(\phi(p)) \tag{2.22}$$

for any $f \in \mathcal{F}(M)$. We can assume that $\phi(p) = 0$. Using the identity from Lemma 2.2.10 one can write the function $g = f \circ \phi^{-1}$ as

$$f(\phi^{-1}(y)) = f(\phi^{-1}(0)) + \sum_{i=1}^n g_i(y) y^i, \tag{2.23}$$

which can be written as

$$f(q) = f(p) + \sum_{i=1}^n (g_i \circ \phi)(q) f_i(q), \tag{2.24}$$

or

$$f = f(p) + \sum_{i=1}^n (g_i \circ \phi) f_i, \tag{2.25}$$

Here, $q = \phi^{-1}(y)$ and $y \in B_0 \subset \mathbb{R}^n$, i.e., y is in a neighborhood of $\phi(p)$. With this representation and Theorem 2.2.5

$$x(f) = \sum_{i=1}^n x(g_i \circ \phi) f_i(p) + \sum_{i=1}^n (g_i \circ \phi)(p) x(f_i). \tag{2.26}$$

The first term in the above expression vanishes due to $\phi(p) = 0$, i.e., $f_i(p) = 0$. Using the property in Lemma 2.2.10, we can rewrite

$$(g_i \circ \phi)(p) = g_i(0) = \frac{\partial g}{\partial x^i}(0) = \frac{\partial (f \circ \phi^{-1})}{\partial x^i}(0). \tag{2.27}$$

Therefore, Eq. (2.26) is equivalent to Eq. (2.22). □

Now that we have discussed local properties of tangent vectors, we can move on and build on this to define global concepts, such as vector fields, Lie brackets, tensor fields etc.

2.3 Vector fields and Lie bracket

As usual, M denotes an n-dimensional $\mathcal{C}^k$-manifold throughout this section, where we will set $k = \infty$ for all practical purposes.

Definition 2.3.1. *A **vector field** X on M is a mapping that assigns each point $p \in M$ a tangent vector $X_p \in T_pM$ in the corresponding (local) tangent space:*

$$X : M \to TM \tag{2.28}$$
$$p \mapsto X_p. \tag{2.29}$$

X is a smooth ($\mathcal{C}^k$) vector field if for all $f \in \mathcal{F}(M)$ the real-valued function X(f), defined by $X(f)(p) = X_p(f)$, is $\mathcal{C}^k$. We denote the linear space of all smooth vector fields on M by $\mathcal{X}(M)$.

One example of a vector field is the mapping $\partial_i : p \mapsto \partial_i|_p$ that maps every point $p \in M$ onto the i-th coordinate basis vector $\partial_i|_p$ in T_pM for a given chart ϕ containing p. We call the $\{\partial_i | i = 1, \ldots, n\}$ the **coordinate vector fields** with respect to the chart ϕ.

Components of a vector field. Let $U \subset M$ be an open neighborhood of $p \in M$, and $(x^1, \ldots, x^n)$ be the local coordinates on U. Note that according to Theorem 2.2.11, any vector field X has a unique local representation

$$X_p = X^i(p)\partial_i|_p. \tag{2.30}$$

Definition 2.3.2. *The n functions $X^i : U \subset M \to \mathbb{R}$ $(i = 1, \ldots, n)$ are called **components** or **coordinate vector fields** of X with respect to the local coordinates $(x^1, \ldots, x^n)$ on U.*

One could have chosen different coordinates on U in Eq. (2.30), and so one may wonder how to relate the unique local representations of a vector field for different coordinate choices?

Change of coordinates. Let us start by working out the transformations for the coordinate vector fields. We assume that in addition $(\bar{U}, \bar{\phi})$ represents another overlapping chart with (U, ϕ) on M with coordinates $(\bar{x}^1, \ldots, \bar{x}^n)$, which gives rise to the change of coordinates $\bar{\phi} \circ \phi^{-1}$ on $U \cap \bar{U}$: $(x^1, \ldots, x^n) \mapsto (\bar{x}^1(x^i), \ldots, \bar{x}^n(x^i))$ (cf. Def. 2.1.1). Starting from Def. 2.2.9 one can write:

$$\partial_i|_p(f) = \frac{\partial(f\circ\phi^{-1})}{\partial x^i}(\phi(p)) \tag{2.31}$$
$$= \frac{\partial(f\circ\bar\phi^{-1}\circ\bar\phi\circ\phi^{-1})}{\partial x^i}(\phi(p)) \tag{2.32}$$
$$= \sum_{j=1}^{n}\frac{\partial(f\circ\bar\phi^{-1})}{\partial\bar x^j}\frac{\partial(\bar\phi\circ\phi^{-1})^j}{\partial x^i}(\phi(p)) \tag{2.33}$$
$$= \frac{\partial\bar x^j}{\partial x^i}\bar\partial_j|_p(f). \tag{2.34}$$

Here, we have used the usual chain rule for functions in $\mathbb{R}^n$. Analogously, one can derive

$$\bar\partial_i|_p = \frac{\partial x^j}{\partial\bar x^i}\partial_j|_p, \tag{2.35}$$

and conclude that the matrices $\partial\bar x^j/\partial x^i$ and $\partial x^j/\partial\bar x^i$ are inverse to each other.

Exercise 2.3.3. *Show that the corresponding transformations for the components of a vector field X are given by*

$$\bar X^i = \frac{\partial\bar x^i}{\partial x^j}X^j. \tag{2.36}$$

Given vector fields $X, Y\in\mathcal{X}(M)$ and $f\in\mathcal{F}(M)$ one can create new scalar and vector fields. For example, the operation $X(f)$ defined by $Xf(p) = X_p(f)$ gives rise to a new scalar function $g = X(f)\in\mathcal{F}(M)$, onto which one can again apply a vector field Y. Note, however, that the resulting expression XY, defined by $XY(f) = X(Y(f))$ is not a vector field as it does not satisfy the product rule in Def. 2.2.2. However, one can construct a vector field by subtracting YX:

Theorem 2.3.4. *Let $X, Y\in\mathcal{X}(M)$. The mapping $[X,Y]|_p : \mathcal{F}(M)\to\mathbb{R}$ defined by $[X,Y]|_p(f)\equiv XYf(p) - YXf(p)$ is a tangent vector in T_pM.*

Proof. We need to check that conditions (i) and (ii) in Def. 2.2.2 are satisfied at every $p\in M$. Let $f, g\in\mathcal{F}(M)$, $\lambda,\mu\in\mathbb{R}$. Using these properties for X and Y individually, one obtains:

$$XY(\lambda f+\mu g)(p) = X(\lambda Y(f)+\mu Y(g))(p) = X_p(\lambda Y(f)+\mu Y(g)) \tag{2.37}$$
$$= \lambda X_pY(f)+\mu X_pY(g) \tag{2.38}$$
$$= \lambda XYf(p)+\mu XYg(p), \tag{2.39}$$

and analogously for the second term by renaming $X\leftrightarrow Y$. Furthermore, applying the product rule (ii) first on the inner and then on the outer field,

$$XY(fg)(p) - YX(fg)(p) = X(Y(fg))(p) - Y(X(fg))(p) \tag{2.40}$$
$$= X_p(Y(f)g) + X_p(fY(g)) - Y_p(X(f)g) - Y_p(fX(g)) \tag{2.41}$$
$$= X_p(Y(f))g(p) + Y_p(f)X_p(g) \tag{2.42}$$
$$+X_p(f)Y_p(g) + f(p)X_p(Y(g)) \tag{2.43}$$
$$-Y_p(X(f))g(p) - X_p(f)Y_p(g) \tag{2.44}$$
$$-Y_p(f)X_p(g) - f(p)Y_p(X(g)) \tag{2.45}$$
$$= XY(f)(p)g(p) + f(p)XY(g)(p) \tag{2.46}$$
$$YX(f)(p)g(p) - f(p)YX(g)(p) \tag{2.47}$$

□

This justifies the following definition:

Definition 2.3.5. *Let $X, Y \in \mathcal{X}(M)$. The smooth vector field $[X, Y] \equiv XY - YX$ is called the* ***Lie bracket****.*

From this definition, one can immediately conclude the following properties of the Lie bracket:

$$[X, Y] = -[Y, X] \tag{2.48}$$

$$[X + Y, Z] = [X, Z] + [Y, Z] \tag{2.49}$$

$$[X, Y + Z] = [X, Y] + [X, Z] \tag{2.50}$$

$$[\lambda X, \mu Y] = \lambda\mu[X, Y], \tag{2.51}$$

where $X, Y, Z \in \mathcal{X}(M)$, $\lambda, \mu \in \mathbb{R}$.

Exercise 2.3.6. *Let $X, Y \in \mathcal{X}(M)$ and $f, g \in \mathcal{F}(M)$. Show that*

$$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X, \tag{2.52}$$

and that the cyclic ***Jacobi identity*** *holds:*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \tag{2.53}$$

Theorem 2.3.7. *Let $\phi : U \subset M \to \mathbb{R}^n$ be a chart of M. The Lie bracket vanishes for any pair of coordinate vector fields on U with respect to ϕ, $[\partial_i, \partial_j] = 0$.*

Proof. Let ∂_i and ∂_j be two arbitrary coordinate vector fields on U induced by ϕ. Then according to Def. 2.2.9 one finds

$$[\partial_i, \partial_j]\big|_p(f) = \frac{\partial}{\partial x^i}\left\{\frac{\partial(f \circ \phi^{-1})}{\partial x^j}(\phi(\cdot)) \circ \phi^{-1}\right\}(\phi(p)) - \frac{\partial}{\partial x^j}\left\{\frac{\partial(f \circ \phi^{-1})}{\partial x^i}(\phi(\cdot)) \circ \phi^{-1}\right\}(\phi(p)) \tag{2.54}$$

$$= \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}(f \circ \phi^{-1})\bigg|_{\phi(p)} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}(f \circ \phi^{-1})\bigg|_{\phi(p)} = 0, \tag{2.55}$$

since the order of differentiation does not matter for smooth functions in $\mathbb{R}^n$. □

Theorem 2.3.8. ***(Coordinate representation of the Lie bracket)*** *Let $X, Y \in \mathcal{X}(M)$, with $X = X^i\partial_i$ and $Y = Y^i\partial_i$ on $U \subset M$ for a given chart $\phi : U \to \mathbb{R}^n$. Then*

$$[X, Y] = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}(X^i\partial_i Y^j - Y^i\partial_i X^j)\right)\partial_j. \tag{2.56}$$

Proof. Using the properties in Eq. (2.52) one finds with the help of Eqs. (2.49), (2.50) and Theorem 2.3.7 that

$$[X, Y] = \sum_{i,k=1}^{n}[X^i\partial_i, Y^k\partial_k] = \sum_{i,k=1}^{n} X^i\partial_i(Y^k)\partial_k - \sum_{i,k=1}^{n} Y^k\partial_k(X^i)\partial_i \tag{2.57}$$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n}(X^i\partial_i Y^j - Y^i\partial_i X^j)\right)\partial_j. \tag{2.58}$$

□

Exercise 2.3.9. *Consider $M = \mathbb{R}^2$ with Cartesian coordinates (x_1, x_2).*

(a) Compute the Lie brackets of the following vector fields:

(i) $x_1\partial_2 - x_2\partial_1$ and $x_1\partial_1 + x_2\partial_2$.

(ii) $X(x_1, x_2) = (x_2, -\sin x_1)$ and $Y(x_1, x_2) = (x_2, \sin x_1)$.

(b) Find two linearly independent vector fields X and Y that are non-vanishing everywhere and which do not commute, i.e., $[X, Y] \neq 0$.

2.4 Tensor fields

Since the local tangent space of a manifold is a linear space, we can consider its **dual space** or **cotangent space** T_p^*M of all covectors with respect to T_pM. This local property then allows us to define the concept of covector and tensor fields on manifolds. In constructing those, we again follow the recipe of making use of local definitions from (multi-)linear algebra and generalizing them globally across the manifold by introducing "fields". As examples of a tensor field, we shall discuss the metric and the curvature tensor in later sections, concepts that are of fundamental importance for general relativity. As usual, M denotes an n-dimensional $\mathcal{C}^k$-manifold throughout this section, where we will set $k = \infty$ for all practical purposes.

Remember from linear algebra that the dual space V^* of the vector space V is the linear space of all linear mappings (covectors) $\omega : V \to \mathbb{R}$. Note that for any basis $v_1, \ldots, v_n$ of V there is a corresponding dual basis $\alpha_1, \ldots, \alpha_n$ of V^* with the property

$$\alpha_i(v_j) = \delta^i_j. \tag{2.59}$$

Definition 2.4.1. *A **covector field** or **one-form** ω on M assigns every $p \in M$ a covector $\omega_p \in T_p^*M$,*

$$\omega : M \to T^*M \tag{2.60}$$

$$p \mapsto \omega_p, \tag{2.61}$$

*where $T^*M = \bigcup_{p\in M} T_p^*M$ is the **cotangent bundle**. ω is a smooth ($\mathcal{C}^k$) covector field if for every smooth vector field X on M the real-valued function $p \mapsto \omega_p(X_p)$ is smooth. The linear space of all smooth covector fields on M is denoted by $\mathcal{X}^*(M)$.*

Definition 2.4.2. *Let $f \in \mathcal{F}(M)$ be a smooth function on an open set $U \subset M$. For arbitrary $p \in U$ and $v \in T_pM$ we define*

$$(\mathrm{d}f)_p(v) \equiv v(f). \tag{2.62}$$

*This gives rise to a linear mapping $(\mathrm{d}f)_p : T_pM \to \mathbb{R}$, i.e., $(\mathrm{d}f)_p \in T_p^*M$. We call $(\mathrm{d}f)_p$ the **differential** of f at the point p. Furthermore, one can define the smooth covector field*

$$(\mathrm{d}f) : p \mapsto (\mathrm{d}f)_p. \tag{2.63}$$

Basis covector fields. Let $(x^1, \ldots, x^n)$ be a coordinate system on $U \subset M$ with chart ϕ. According to Def. 2.2.9, we then have

$$(\mathrm{d}f)_p\,(\partial_i|_p) = \frac{\partial(f \circ \phi^{-1})}{\partial x^i}(\phi(p)). \tag{2.64}$$

Observe that for the component function $x^i : p \mapsto x^i(\phi(p))$ this reduces to

$$(\mathrm{d}x^i)_p\,(\partial_j|_p) = \frac{\partial x^i}{\partial x^j}(\phi(p)) = \delta^i_j. \tag{2.65}$$

Comparing with Eq. (2.59) we conclude that the differentials $\{(\mathrm{d}x^1)_p, \ldots, (\mathrm{d}x^n)_p\}$ of the component functions x^i with respect to a chart ϕ form a basis of T_p^*M, which is dual to the basis $\{\partial_1|_p, \ldots, \partial_n|_p\}$ of T_pM. The construction $(dx^i) : p \in M \mapsto (dx^i)_p \in T_p^*M$ gives rise to smooth **basis covector fields** $\{(\mathrm{d}x^i)\}$ of T^*M on the chart (U, ϕ).

Coordinate representation. Since $\{(\mathrm{d}x^i)\}$ are basis covector fields on the corresponding chart (ϕ, U), any covector field $\omega \in \mathcal{X}^*(M)$ can be written as

$$\omega = \omega_i \mathrm{d}x^i, \tag{2.66}$$

where ω_i are the components of ω with respect to the chart (U, ϕ). Specifically in the case of differentials, any $(\mathrm{d}f)_p$ can be written as

$$(\mathrm{d}f)_p = \lambda_i (\mathrm{d}x^i)_p, \tag{2.67}$$

which implies

$$(\mathrm{d}f)_p(\partial_i|_p) = \lambda_j (\mathrm{d}x^j)_p(\partial_i|_p) = \lambda_i. \tag{2.68}$$

Comparing with Eq. (2.64), we can thus write

$$(\mathrm{d}f)_p = \frac{\partial(f \circ \phi^{-1})}{\partial x^i}(\phi(p))(\mathrm{d}x^i)_p, \tag{2.69}$$

which is a generalized expression for differentials on manifolds. Choosing, as a special case, $(U, \phi) = (\mathbb{R}^n, \mathrm{Id})$, where Id is the identity map on $\mathbb{R}^n$, this reduces to the well-known expression for the differential of a function f on $\mathbb{R}^n$:

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i. \tag{2.70}$$

Change of coordinates. Given two overlapping charts (U, ϕ) and $(\bar{U}, \bar{\phi})$ on M with coordinates $x_1, \ldots, x_n$ and $\bar{x}_1, \ldots, \bar{x}_n$, respectively, how do the components of a covector field transform under a change of coordinates $\bar{\phi} \circ \phi^{-1}$? According to Eq. (2.66) the covector basis fields on $\bar{U}$ can be expressed as a linear combination of those on U, $\mathrm{d}\bar{x}^i = \omega^i_k \mathrm{d}x^k$, for some coefficients ω^i_k. Therefore,

$$\mathrm{d}\bar{x}^i(\partial_j) = \omega^i_k \mathrm{d}x^k(\partial_j) = \omega^i_j. \tag{2.71}$$

Using Eq. (2.34) one also finds that

$$\mathrm{d}\bar{x}^i(\partial_j) = \mathrm{d}\bar{x}^i\left(\frac{\partial \bar{x}^k}{\partial x^j}\bar{\partial}_k\right) = \frac{\partial \bar{x}^k}{\partial x^j}\mathrm{d}\bar{x}^i(\bar{\partial}_k) = \frac{\partial \bar{x}^i}{\partial x^j}. \tag{2.72}$$

We thus conclude that the basis covector fields transform according to

$$\mathrm{d}\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j}\mathrm{d}x^j. \tag{2.73}$$

Exercise 2.4.3. *Show that the components of a covector field $\omega = \omega_i \mathrm{d}x^i$ transform under a change of coordinates according to*

$$\bar{\omega}_i = \frac{\partial x^j}{\partial \bar{x}^i}\omega_j. \tag{2.74}$$

Tensor fields. We will now show that vector fields and covector fields are just simple examples of a more general concept—tensor fields. Such tensor fields will later prove useful to describe important properties of spacetime, which are central to general relativity, such as the metric, spactime curvature etc. Remember from (multi-)linear algebra that given an n-dimensional vector space V with dual space V^*, a multilinear function $T : V^{*r} \times V^s \to \mathbb{R}$ $(r, s > 0)$ is called a **tensor** of contravariant rank r and covariant rank s (or (r, s)-tensor for short); 'multilinear' means that T is linear in all its arguments, i.e., it has the following properties:

$$\begin{aligned} &T(\alpha^1, \ldots, \alpha^{i-1}, \lambda\alpha + \mu\beta, \alpha^{i+1}, \ldots, \alpha^r, v_1, \ldots, v_s) = \\ &\quad \lambda T(\alpha^1, \ldots, \alpha^{i-1}, \alpha, \alpha^{i+1}, \ldots, \alpha^r, v_1, \ldots, v_s) + \mu T(\alpha^1, \ldots, \alpha^{i-1}, \beta, \alpha^{i+1}, \ldots, \alpha^r, v_1, \ldots, v_s) \end{aligned} \tag{2.75}$$

and

$$\begin{aligned} &T(\alpha^1, \ldots, \alpha^r, v_1, \ldots, v_{k-1}, \lambda v + \mu w, v_{k+1}, \ldots, v_s) = \\ &\quad \lambda T(\alpha^1, \ldots, \alpha^r, v_1, \ldots, v_{k-1}, v, v_{k+1}, \ldots, v_s) + \mu T(\alpha^1, \ldots, \alpha^r, v_1, \ldots, v_{k-1}, w, v_{k+1}, \ldots v_s), \end{aligned} \tag{2.76}$$

for any $\lambda, \mu \in \mathbb{R}$. The special class of (r, s)-tensors defined by

$$T(\beta^1, \ldots, \beta^r, w_1, \ldots, w_s) = \beta^1(v_1)\ldots\beta^r(v_r)\alpha^1(w_1)\ldots\alpha^s(w_s) \tag{2.77}$$

for some fixed $v_1, \ldots, v_r \in V$ and $\alpha^1, \ldots, \alpha^s \in V^*$ are called **simple tensors**, and are denoted by

$$T = v_1 \otimes \ldots \otimes v_r \otimes \alpha^1 \otimes \ldots \otimes \alpha^s. \tag{2.78}$$

One can show that the n^{r+s} simple tensors

$$\left\{v_{i_1} \otimes \ldots \otimes v_{i_r} \otimes \alpha^{j_1} \otimes \ldots \otimes \alpha^{j_s} \middle| i_1, \ldots, i_r, j_1, \ldots, j_s = 1, \ldots, n\right\} \tag{2.79}$$

where $v_1, \ldots, v_n$ is a basis of V and $\alpha^1, \ldots, \alpha^n$ its dual basis of V^*, form a basis for the linear space of all (r, s)-tensors; this space is denoted by V^r_s and has thus dimension n^{r+s}.

Using this as a local concept, let us now apply it to manifolds and define global tensor fields based on the local (linear) tangent spaces we constructed earlier.

Definition 2.4.4. *Let $T_pM^r_s$ denote the linear space of tensors of rank (r,s) on T_pM (contravariant of rank r, covariant of rank s). The union of all local tensor spaces is called a* ***tensor bundle****, $TM^r_s \equiv \bigcup_{p\in M} T_pM^r_s$. A* ***tensor field of type*** (r, s) *on M is a mapping that assigns every $p \in M$ a tensor $T_p \in T_pM^r_s$:*

$$T : M \to TM^r_s \tag{2.80}$$

$$p \mapsto T_p. \tag{2.81}$$

T is a smooth ($\mathcal{C}^k$) tensor field if for any smooth vectorfields $X_1, \ldots, X_s$ and smooth covector fields $\omega_1, \ldots, \omega_r$ the real-valued function

$$p \mapsto T_p(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) \tag{2.82}$$

is smooth. We denote by $\mathcal{T}^r_s(M)$ the linear space of all smooth (r, s)-tensor fields on M.

Some remarks:

- One can show that the spaces of all smooth vector fields $\mathcal{X}(M)$ and covector fields $\mathcal{X}^*(M)$ on M can be identified with $\mathcal{T}_1^0(M)$ and $\mathcal{T}_0^1(M)$, respectively (they are isomorphic to each other). In this sense, vector fields and covector fields are just special cases of tensor fields.

- Algebraic operations on tensor fields are defined pointwise, e.g.,

$$(T+S)_p = T_p + S_p, \quad T, S \in T_pM^r_s, \tag{2.83}$$
$$(fT)_p = f(p)T_p, \quad T \in T_pM^r_s,\ f \in \mathcal{F}(M). \tag{2.84}$$

- In a chart (U,ϕ) on M with local coordinates $x^1,\ldots,x^n$, any (r,s)-tensor field has a **component representation** in terms of the local basis tensor fields of TM^r_s:

$$T = T^{i_1\ldots i_r}_{j_1\ldots j_s}\partial_{i_1}\otimes\ldots\otimes\partial_{i_r}\otimes \mathrm{d}x^{j_1}\otimes\ldots\otimes\mathrm{d}x^{j_s}, \tag{2.85}$$

  where

$$T^{i_1\ldots i_r}_{j_1\ldots j_s} = T(\mathrm{d}x^{i_1},\ldots,\mathrm{d}x^{i_r},\partial_{j_1},\ldots,\partial_{j_s}), \tag{2.86}$$

  are the components of T in the chart ϕ. This follows from the local representation in terms of tensor basis functions on $T_pM^r_s$ discussed above, together with the extension of local basis functions $\{\partial_i|_p\}$ and $\{(\mathrm{d}x^i)_p\}$ to global ones on a chart (U,ϕ) (see above).

Change of coordinates. Given the expansion of tensors in terms of their local coordinate components, one may wonder how these components transform under a change of coordinates. The following identity will prove useful for practical computations:

Theorem 2.4.5. *Let $T \in \mathcal{T}^r_s(M)$ be a smooth tensor field on M, and (U,ϕ) and $(\bar U,\bar\phi)$ two overlapping charts with coordinates $x^1,\ldots,x^n$ and $\bar x^1,\ldots,\bar x^n$, respectively. Under a **change of coordinates** in the overlapping region $U\cap\bar U$, the components of T then transform according to*

$$\bar T^{i_1\ldots i_r}_{j_1\ldots j_s} = \frac{\partial\bar x^{i_1}}{\partial x^{l_1}}\cdots\frac{\partial\bar x^{i_r}}{\partial x^{l_r}}\frac{\partial x^{k_1}}{\partial\bar x^{j_1}}\cdots\frac{\partial x^{k_s}}{\partial\bar x^{j_s}}T^{l_1\ldots l_r}_{k_1\ldots k_s}. \tag{2.87}$$

Exercise 2.4.6. *Proof Theorem 2.4.5 by making use of Eq. (2.86) and the transformation properties Eqs. (2.35) and (2.73).*

Tensor contractions. Given a tensor, one can construct tensors of lower rank by 'contractions':

Definition 2.4.7. *Given a tensor field $T \in \mathcal{T}^r_s(M)$ on M and $l \in \{1,\ldots,r\}$, $m\in\{1,\ldots,s\}$. Furthermore, let $\{\partial_i\}$ and $\{\mathrm{d}x^i\}$ denote the coordinate basis vector and covector fields on a chart (U,ϕ). The contraction C^l_mT of T is defined as the tensor $C^l_mT\in\mathcal{T}^{r-1}_{s-1}(M)$ by*

$$C^l_mT(\omega^1,\ldots,\omega^{l-1},\omega^{l+1},\ldots,\omega^r,X_1,\ldots,X_{m-1},X_{m+1},\ldots,X_s) = \\ T(\omega^1,\ldots,\omega^{l-1},\mathrm{d}x^k,\omega^{l+1},\ldots,\omega^r,X_1,\ldots,X_{m-1},\partial_k,X_{m+1},\ldots,X_s), \tag{2.88}$$

where $X_1,\ldots,X_s\in\mathcal{X}(M)$ and $\omega^1,\ldots,\omega^r\in\mathcal{X}^*(M)$.

Note the sum over k in the preceding definition. Comparing to Eq. (2.86), we conclude that the components of the contracted tensor are given by

$$(C^l_mT)^{i_1\ldots i_{l-1}i_{l+1}\ldots i_r}_{j_1\ldots j_{m-1}j_{m+1}\ldots j_s} = T^{i_1\ldots i_{l-1}ki_{l+1}\ldots i_r}_{j_1\ldots j_{m-1}kj_{m+1}\ldots j_s}. \tag{2.89}$$

Exercise 2.4.8. *Show that the contraction of tensors is independent of the basis vector and covector fields used.*

2.5 (Semi-)Riemannian manifolds, the metric, and the definition of spacetime

Let M be an n-dimensional $\mathcal{C}^k$-manifold, with $k = \infty$ for all practical purposes.

Definition 2.5.1. *A* ***pseudo-Riemannian metric*** *on M is a tensor field $g \in \mathcal{T}_2^0(M)$ with the properties:*

(i) It is symmetric, i.e., $g(X,Y) = g(Y,X)$ for all vector fields $X, Y \in \mathcal{X}(M)$.

(ii) For all $p \in M$, $g_p \in T_pM_2^0$ is a non-degenerate bilinear form on T_pM, i.e., $g_p(x,y) = 0$ for all $x \in T_pM$ if and only if $y = 0$.

Such a tensor field is called a ***Riemannian metric*** *if, in addition, g_p is positive definite for all $p \in M$, i.e., $g_p(x,x) \geq 0$ for all $x \in T_pM$ and $g_p(x,x) = 0$ if and only if $x = 0$ (that is, g_p defines an inner product on T_pM). A* ***(pseudo-)Riemannian manifold*** *is a smooth manifold M together with a (pseudo-)Riemannian metric g.*

Having a (pseudo-)Riemannian metric g on M is extremely useful. Let us discuss some properties of g and what it allows one to do. In the following, let (U, ϕ) be a chart of M with coordinates $x_1, \ldots, x_n$ and local basis vector fields $\partial_1, \ldots, \partial_n$ and basis covector fields $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$.

Coordinate representation. According to Eqs. (2.85) and (2.86), g has a coordinate representation

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j, \tag{2.90}$$

where

$$g_{ij} = g(\partial_i, \partial_j). \tag{2.91}$$

The metric is often written in terms of the **line element**,

$$ds^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{2.92}$$

which measures infinitesimal distances on M (see below). Here, we also introduced the frequently used short form

$$\mathrm{d}x^i\mathrm{d}x^j = \frac{1}{2}(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i). \tag{2.93}$$

Raising and lowering indices. Observe that g assigns every vector field a corresponding covector field and vice versa. One can show that the mapping between $\mathcal{X}(M)$ and $\mathcal{X}^*(M)$ g gives rise to,

$$g^\flat : \mathcal{X}(M) \;\to\; \mathcal{X}^*(M) \tag{2.94}$$

$$X \;\mapsto\; X^\flat \equiv g(X, \cdot), \tag{2.95}$$

is bijective. Writing in terms of basis fields, $X = X^i\partial_i$ and $X^\flat = X_i\mathrm{d}x^i$, we find that

$$X^\flat(Y) = X_i\mathrm{d}x^i(Y^j\partial_j) = X_iY^j\mathrm{d}x^i(\partial_j) = X_iY^i, \tag{2.96}$$

and

$$X^\flat(Y) = g(X,Y) = g(X^i\partial_i, Y^j\partial_j) = X^iY^jg(\partial_i, \partial_j) = g_{ij}X^iY^j. \tag{2.97}$$

Therefore, we conclude that the coordinate components of the corresponding covector field $X^\flat$ are given by

$$X_i = g_{ij} X^j. \tag{2.98}$$

In order to find the inverse of the mapping $g^\sharp = (g^\flat)^{-1}$, we guess that it is of the form

$$\begin{aligned} g^\sharp : \mathcal{X}^*(M) &\rightarrow \mathcal{X}(M) && (2.99)\\ \omega &\mapsto \omega^\sharp, && (2.100) \end{aligned}$$

where $\omega^\sharp(f) \equiv g^{-1}(\omega, \mathrm{d}f)$ for any $f \in \mathcal{F}(M)$ and some $g^{-1} \in \mathcal{T}_0^2(M)$ with components $g^{-1}(\mathrm{d}x^i, \mathrm{d}x^j) = g^{ij}$ still to be specified. One can write $\omega^\sharp = X^i \partial_i$ for some components X^i, as well as $\omega = X_i \mathrm{d}x^i$ for given components X_i. Using the definition of the basis vector fields Def. (2.2.9), we find that

$$\omega^\sharp(f) = X^i \partial_i(f) = X^i \frac{\partial (f \circ \phi^{-1})}{\partial x^i}(\phi(\cdot)) \tag{2.101}$$

and

$$\begin{aligned} \omega^\sharp(f) &= g^{-1}(\omega, \mathrm{d}f) = g^{-1}\left(X_j \mathrm{d}x^j, \frac{\partial (f \circ \phi^{-1})}{\partial x^i}(\phi(\cdot))\mathrm{d}x^i\right) && (2.102)\\ &= X_j \frac{\partial (f \circ \phi^{-1})}{\partial x^i}(\phi(\cdot)) g^{-1}(\mathrm{d}x^j, \mathrm{d}x^i) && (2.103)\\ &= g^{ij} X_j \frac{\partial (f \circ \phi^{-1})}{\partial x^i}(\phi(\cdot)), && (2.104) \end{aligned}$$

where we have used Eq. (2.69). Comparing these two expressions, we conclude that the components of the vector field X corresponding to ω are given by

$$X^i = g^{ij} X_j. \tag{2.105}$$

And since applying $g^\flat$ and then $g^\sharp$ has to provide the same vector field, $g^\sharp \circ g^\flat = \mathrm{Id}$, substituting Eq. (2.98) into Eq. (2.105), we find that g^{-1} is defined by the components g^{ij} that satisfy

$$g^{ik} g_{kj} = \delta^i_j. \tag{2.106}$$

Thinking in terms of matrix components, g^{ij} are thus the inverse matrix components to the metric components g_{ij}. This contravariant tensor $g^{-1} \in \mathcal{T}_0^2(M)$ associated with g (often also simply referred to as g) is called **contravariant metric tensor** (or **contravariant metric** for short).

Transforming vector fields into covector fields and vice versa using the metric or the contravariant metric can immediately be extended to any rank k of an (r, s)-tensor field:

$$\begin{aligned} g^{\flat k} : \mathcal{T}_s^r(M) &\rightarrow \mathcal{T}_{s-1}^{r+1}(M) && (2.107)\\ T &\mapsto T^{\flat k}, && (2.108) \end{aligned}$$

with

$$T^{\flat k}(\omega^1, \ldots, \omega^r, X_1, \ldots, X_{k-1}, \omega^{r+1}, X_{k+1}, \ldots, X_s) = T(\omega^1, \ldots, \omega^r, X_1, \ldots, X_{k-1}, (\omega^{r+1})^\sharp, X_{k+1}, \ldots, X_s), \tag{2.109}$$

and

$$g^{\sharp k} : \mathcal{T}_s^r(M) \rightarrow \mathcal{T}_{s+1}^{r-1}(M) \tag{2.110}$$
$$T \mapsto T^{\sharp k}, \tag{2.111}$$

with

$$T^{\sharp k}(\omega^1, \ldots, \omega^{k-1}, X_{s+1}, \omega^{k+1}, \ldots, \omega^r, X_1, \ldots, X_s) = T(\omega^1, \ldots, \omega^{k-1}, (X_{s+1})^\flat, \omega^{k+1}, \ldots, \omega_r, X_1, \ldots, X_s) \tag{2.112}$$

and $X_1, \ldots, X_{s+1} \in \mathcal{X}(M)$, $\omega^1, \ldots, \omega^{r+1} \in \mathcal{X}^*(M)$. Employing the rules (2.98) and (2.105), we obtain the generalized transformations for the tensor coordinate components:

$$T^{i_1 \ldots i_{k-1}}{}_{i_k}{}^{i_{k+1} \ldots i_r}{}_{j_1 \ldots j_s} = g_{i_k m} T^{i_1 \ldots i_{k-1} m i_{k+1} \ldots i_r}{}_{j_1 \ldots j_s}, \tag{2.113}$$

as an example for 'lowering one index', and

$$T^{i_1 \ldots i_r}{}_{j_1 \ldots j_{k-1}}{}^{j_k}{}_{j_{k+1} \ldots j_s} = g^{j_k m} T^{i_1 \ldots i_r}{}_{j_1 \ldots j_{k-1} m j_{k+1} \ldots j_s}, \tag{2.114}$$

as an example of 'raising one index'. Note that we have horizontally separated the contravariant and covariant indices here to clearly indicate which index has been raised or lowered. Note that Eqs. (2.113) and (2.114) are sometimes used to *define* tenors, as the reader may remember from introductory physics courses. Here, we have taken a different approach and *derived* these transformation rules for tensor components from fundamental geometric concepts of (pseudo-)Riemannian manifolds.

Canonical form and Lorentzian manifolds. We recall a basic theorem from linear algebra, adapted to our context of tangent spaces of a (pseudo-)Riemannian manifold:

Theorem 2.5.2. *For any symmetric bilinear form g_p on an n-dimensional linear space T_pM, there exists a basis $v_1, \ldots, v_n$ of T_pM, such that the components g_{ij} of g with respect to that basis are of the* ***canonical form***

$$g_{ij} = \begin{pmatrix} -1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ddots & \ddots & & & & & & \vdots \\ \vdots & \ddots & -1 & \ddots & & & & & \vdots \\ \vdots & & \ddots & 1 & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & & \ddots & 1 & \ddots & & \vdots \\ \vdots & & & & & \ddots & 0 & \ddots & \vdots \\ \vdots & & & & & & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & 0 \end{pmatrix} \tag{2.115}$$

The number of occurrences of -1 and of $+1$ on the diagonal are independent of the choice of the original basis in which the components of g_p are given.

The representation of g_p in Eq. (2.115) is usually abbreviated by

$$g_{ij} = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1, 0, \ldots, 0). \tag{2.116}$$

Note that these diagonal elements are the eigenvalues of g_p. For (pseudo-)Riemannian manifolds we excluded degenerate forms (cf. Def. 2.5.1), so that the diagonal elements will only be -1 or 1 in practice. Also note that for Riemannian manifolds, $g_{ij} = \mathrm{diag}(1, \ldots, 1)$; pseudo-Riemannian manifolds have indefinite metrics, i.e., metrics with both -1's and $+1$'s in the canonical form. This gives rise to a special class of pseudo-Riemannian manifolds:

Definition 2.5.3. *A pseudo-Riemannian manifold M whose metric g has a canonical form*

$$g_{ij} = \mathrm{diag}(-1, 1, \ldots, 1) \tag{2.117}$$

at every point $p \in M$ is called a ***Lorentzian manifold****.*

Four-dimensional Lorentzian manifolds thus have tangent spaces endowed with the structure of Minkowski space, i.e., the metric can be locally transformed into the Minkowski metric $\eta_{ij} = \mathrm{diag}(-1, 1, 1, 1)$. If we adopt the notion of a tangent space as locally describing the manifold, the tangent spaces being equivalent (isomorphic) to Minkowski space thus has important implications in the context of the equivalence principle (Sec. 1.2). The structure of Lorentzian manifolds appears to encode Einstein's equivalence principle: locally and in a special frame (a freely-falling frame–to be specified later) spacetime reduces to Minkowski space and the laws of special relativity apply. We will formalize this notion further in the next chapters, but this already hints at Lorentzian manifolds being a candidate for spacetime in general relativity.

One can generalize the concept of timelike, spacelike, and null vectors in Minkowski space to Lorentzian manifolds in a straightforward way:

Definition 2.5.4. *A tangent vector $x \in T_pM$ of a Lorentzian manifold M is called*

- ***timelike****, if $g_p(x, x) < 0$,*
- ***spacelike****, if $g_p(x, x) > 0$ or $x = 0$,*
- ***null****, if $g_p(x, x) = 0$ and $x \neq 0$.*

Accordingly, any vector field $X \in \mathcal{X}(M)$ on M is called

- ***timelike****, if $g(X, X) < 0$ everywhere,*
- ***spacelike****, if $g(X, X) > 0$ everywhere, or $X = 0$,*
- ***null****, if $g(X, X) = 0$ everywhere and $X \neq 0$.*

Measuring lengths. The metric on Riemannian or Lorentzian manifolds can be used to measure lengths of curves and thus distances on the manifold. Following the previous definition, a curve $c : I \subset \mathbb{R} \to M$ on a Lorentzian manifold is called timelike if $\dot{c}(t) \in T_{c(t)}M$ is timelike for all $t \in I$, spacelike if $\dot{c}(t)$ is spacelike for all t, and null if $\dot{c}(t)$ is null for all t.

Let $\mathrm{d}x$ denote a small vector displacement in T_pM. Its squared length is given by $g_p(\mathrm{d}x, \mathrm{d}x) = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$, which is the squared line element $\mathrm{d}s^2$ (cf. Eq. (2.92)). Taking the absolute value and the square root, we obtain a measure of length, which we can re-express in terms of the tangent vector of a curve, recalling that the tangent vector of a curve c can be written as $\dot{c}(t) = (\mathrm{d}x^i/\mathrm{d}t)\partial_i$:

$$\mathrm{d}s = |g_{ij}\mathrm{d}x^i\mathrm{d}x^j|^{1/2} = \left| g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} \right|^{1/2} \mathrm{d}t = \sqrt{|g(\dot{c},\dot{c})|}\mathrm{d}t. \tag{2.118}$$

This gives rise to the following definition:

Definition 2.5.5. *Let (M, g) be a Riemannian or Lorentzian manifold. The length of a curve $c : I \subset \mathbb{R} \to M$ is defined by*

$$L(c) = \int_I \sqrt{\pm g(\dot{c}(t), \dot{c}(t))}\mathrm{d}t, \tag{2.119}$$

where $\dot{c}(t) \in T_{c(t)}M$ is the tangent vector of c at t. The $\pm$-sign only applies to Lorentzian manifolds, in which case only timelike curves $(-)$ or spacelike curves $(+)$ are considered.

Causality and spacetime. As we already noted, four-dimensional Lorentzian manifolds are a good candidate for spacetime in general relativity. For those to be reasonable physical models of spacetime, however, we need to impose (at least) two more conditions. First, such manifolds need to be orientable and connected (which we shall not discuss here). More interesting in our context is the issue of time-orientability on Lorentzian manifolds, which is intricately linked to causality.

In Newtonian physics causality is imposed by the concept of universal time that always and everywhere moves forward. In special relativity, the speed of light imposes a restriction on which events can influence future events. In order to stay in causal contact, the worldline connecting two events must stay within its forward light cone. If a Lorentzian manifold is to represent spacetime in general relativity, one needs to make sure that causality is respected. It may occur that even when an observer stays within its forward light cone, following a timelike curve on the manifold, that curve may insect itself at some time in the observer's "past". Such curves are called **closed timelike curves**. Such curves may exist because the curvature of the manifold (to be discussed in the next chapters) can significantly 'tilt' the local light cones. One simple example is given by a cylindrical manifold $M = \mathbb{R} \times S^1$ with coordinates $\{t, x\} \in \mathbb{R} \times [0, 1]$, where (t, x) and $(t, x + 1)$ are identified, and metric

$$\mathrm{d}s^2 = \cos(\tau)\mathrm{d}t^2 - \sin(\tau)[\mathrm{d}t\mathrm{d}x + \mathrm{d}x\mathrm{d}t] - \cos(\tau)\mathrm{d}x^2. \tag{2.120}$$

Here, $\tau = \cot^{-1} t$.

Exercise 2.5.6. *Show that*

(a) (M, g) *is a Lorentzian manifold.*

(b) $c : \mathbb{R} \to M$, $c(\lambda) = (\lambda, 0)$ *is timelike for* $\lambda < 0$.

(c) $c : [0, 1] \to M$, $c(\lambda) = (t_0, \lambda)$ *is timelike for* $t_0 > 0$.

Exercise 2.5.6 shows that for $t < 0$, t is the timelike coordinate, i.e., the light cones point in direction of t. However, for $t > 0$, x becomes the timelike coordinate and there are timelike curves $c : [0, 1] \to M$, $c(\lambda) = (t_0, \lambda)$, $t_0 > 0$, that wrap once around the cylinder and self-intersect, i.e., they are closed timelike curves. This means that an observer could travel on the manifold along a timelike curve toward the 'future' and return back to his/her 'past', which violates any notion of causality.

This example motivates the following

Definition 2.5.7. *Let M be a (connected) Lorentzian manifold, and let $\mathcal{C}_{p,\pm} = \{x \in T_pM | g_p(x,x) < 0\}$ denote the two light cones of the tangent space T_pM at $p \in M$, where $+$ and $-$ refer to the future and past light cones, respectively. M is called **time-oriented** if for any timelike vector field $X \in \mathcal{X}(M)$, $g(X,X) < 0$, with $X_p \in \mathcal{C}_{p,+}$ for any $p \in M$ also $X_q \in \mathcal{C}_{q,+}$ for all $q \in M$.*

The preceding example has shown that indeed not all Lorentzian manifolds can be time-oriented. We are now finally ready to define spacetime:

Definition 2.5.8. *A connected, oriented, and time-oriented four-dimensional Lorentzian manifold is called **spacetime**.*

2.6 Covariant derivative

In flat space ($\mathbb{R}^n$) one has the familiar directional derivative of a vector field $Y \in \mathcal{X}(\mathbb{R}^n)$ along some curve c at a point $p \in \mathbb{R}^n$, given by

$$D_x Y = \lim_{h \to 0} \frac{Y(c(h)) - Y(c(0))}{h}, \tag{2.121}$$

with $c(0) = p$ and $\dot{c}(0) = x$. This can be generalized to a derivative of Y along some other vector field $X \in \mathcal{X}(\mathbb{R}^n)$,

$$D_X Y(p) = D_{X_p} Y = X^i \partial_i Y^k e_k, \tag{2.122}$$

using an integral curve c of X at p (i.e., a curve with $\dot{c}(0) = X_p$). From Eq. (2.122) one can show that the directional derivative has the following properties:

- $D_{fX+gY} Z = f D_X Z + g D_Y Z$
- $D_X(\lambda Y + \mu Z) = \lambda D_X Y + \mu D_X Z$
- $D_X(fY) = (Xf)Y + f D_X Y$
- $Z(X \cdot Y) = D_Z X \cdot Y + X \cdot D_Z Y$
- $D_X Y - D_Y X = [X,Y]$

for $X, Y, Z \in \mathcal{X}(\mathbb{R}^n)$, $f, g \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda, \mu \in \mathbb{R}$.

How can one generalize the concept of directional derivatives of vector fields to manifolds? Obviously, the defining expression Eq. (2.121) does not translate to manifolds, as, for example $Y(c(h))$ and $Y(c(0))$ would belong to two different tangent spaces, and can thus not be substracted from each other. Instead, one can employ the above listed properties to *axiomatically define* such an abstract directional derivative on manifolds, as we will do below. In Theorem 2.8.4, however, we will reinterpret the the abstract definition in terms of parallel transport and recover an expression similar to Eq. (2.121).

Definition 2.6.1. *Let M be a smooth manifold, $X, Y, Z \in \mathcal{X}(M)$, $f, g \in \mathcal{F}(M)$, and $\lambda, \mu \in \mathbb{R}$. A mapping $(X,Y) \mapsto \nabla_X Y \in \mathcal{X}(M)$ with the properties*

(i) $\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z$

(ii) $\nabla_X(\lambda Y + \mu Z) = \lambda \nabla_X Y + \mu \nabla_X Z$

(iii) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$

*is called a **connection** on M.*

The name 'connection' is derived from the fact that such a connection gives rise to parallel transporting a vector field along a curve (see Sec. 2.8), and thus provides a means of 'connecting' tangent spaces at different points on the manifold. The other properties of the directional derivative in $\mathbb{R}^n$ are provided by the **fundamental theorem of Riemannian geometry**:

Theorem 2.6.2. *For every (pseudo-)Riemannian manifold (M, g) there exists a unique connection ∇ with the additional properties*

(iv) $Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$ *(the so-called **Ricci identity**),*

(v) $\nabla_X Y - \nabla_Y X = [X, Y]$ *(∇ is **symmetric** or **torsion-free**),*

where $X, Y, Z \in \mathcal{X}(M)$.

Proof. We will show that for a connection satisfying (iv) and (v) the metric g induces an explicit definition of $\nabla_X Y$. Substituting (v),

$$\nabla_X Z - \nabla_Z X = [X, Z], \tag{2.123}$$

into (iv),

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{2.124}$$

one obtains

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_Z X) + g(Y, [X, Z]). \tag{2.125}$$

Analogously, by cyclic permutation,

$$Yg(Z, X) = g(\nabla_Y Z, X) + g(Z, \nabla_X Y) + g(Z, [Y, X]), \tag{2.126}$$

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Y Z) + g(X, [Z, Y]). \tag{2.127}$$

The linear combination (2.125) + (2.126) − (2.127) yields

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \tag{2.128}$$

known as the **Koszul formula**. Note that the right-hand side is independent of ∇—a given number at $p \in M$ for given X, Y, Z at p. Since g is non-degenerate, $\nabla_X Y$ at $p \in M$ is uniquely defined by Z_p. This shows uniqueness of $\nabla_X Y$.

Regarding existence, let us define the vector field $\nabla_X Y$ through the Koszul formula. One can then easily show that the properties properties (i)–(v) are satisfied (see following exercise). □

Exercise 2.6.3. *Assume that $\nabla_X Y$ is defined through the Koszul formula (2.128). Show that $\nabla_X Y$ then satisfies the properties (i)–(v).*

The important conclusion from the proof of Theorem 2.6.2 is that the metric g on (semi-)Riemannian manifolds directly *induces* a (unique) connection with the additional properties (iv)–(v) through the Koszul formula. This leads to the following

Definition 2.6.4. *Let M be a (pseudo-)Riemannian manifold. A connection ∇ on M with the properties $(i) - (v)$ listed above is called the **Levi-Civita connection** or **covariant derivative** of Y with respect to X.*

Remark: The Ricci identity can be shown to be equivalent to what is often referred to as the **“metric compatibility condition”**, namely that the covariant derivative of the metric tensor vanishes. This will be discussed in Sec. 2.9 and Exercise 2.9.7.

2.7 Coordinate representation of the covariant derivative: Christoffel symbols

Let (M, g) be an n-dimensional (pseudo-)Riemannian manifold with Levi-Civita connection ∇.

Definition 2.7.1. *Let (U, ϕ) be a chart of M. The n^3 real-valued functions Γ^i_{jk}, defined by*

$$\nabla_{\partial_j}\partial_k = \Gamma^i_{jk}\partial_i, \tag{2.129}$$

*are called **Christoffel symbols** or **connection coefficients** of the connection ∇ in the given chart ϕ.*

The reason why the Christoffel symbols are also referred to as the connection coefficients is obvious when expressing the covariant derivative in coordinate components with respect to a local chart:

Theorem 2.7.2. *In a local chart (U, ϕ), the coordinate components of the covariant derivative can be written as*

$$(\nabla_X Y)^i = X^j\partial_j Y^i + \Gamma^i_{jk}X^jY^k. \tag{2.130}$$

for $X, Y \in \mathcal{X}(M)$, with $X = X^i\partial_i$ and $Y = Y^i\partial_i$ in the local chart.

Proof. Using the properties of the connection (cf. Def. 2.6.1),

$$\begin{aligned} \nabla_X Y &= \nabla_{X^j\partial_j} Y^i\partial_i = X^j\nabla_{\partial_j} Y^i\partial_i & (2.131)\\ &= X^j(\partial_j(Y^i)\partial_i + X^jY^k\nabla_{\partial_j}\partial_k) & (2.132)\\ &= (X^j\partial_j Y^i + \Gamma^i_{jk}X^jY^k)\partial_i. & (2.133) \end{aligned}$$

□

The important conclusion of Theorem 2.7.2 is that, in local coordinates, the covariant derivative amounts to taking the partial (directional) derivative as in flat space (cf. Eq. (2.122)) plus a correction term that is determined by the Christoffel symbols.

In the following, let us derive a few properties of the Christoffel symbols that will prove useful in practical calculations.

Theorem 2.7.3. *Let Γ^i_{jk} denote the Christoffel symbols of the covariant derivative ∇ on M in a given chart (U, ϕ). Then:*

(i) The Christoffel symbols have the following symmetry:

$$\Gamma^i_{jk} = \Gamma^i_{kj}. \tag{2.134}$$

(ii) The Christoffel symbols can be directly computed from the metric tensor:

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}). \tag{2.135}$$

(iii) Given another overlapping chart $(\bar{U}, \bar{\phi})$*, the Christoffel symbols with respect to* $\bar{\phi}$ *are given by*

$$\bar{\Gamma}^i_{jk} = \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^n} \Gamma^n_{lm} + \frac{\partial^2 x^l}{\partial \bar{x}^j \partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^l}. \tag{2.136}$$

Proof. (i):

$$(\Gamma^i_{jk} - \Gamma^i_{kj})\partial_i = \nabla_{\partial_j}\partial_k - \nabla_{\partial_k}\partial_j = [\partial_j, \partial_k] = 0, \tag{2.137}$$

where we have used property (v) in Theorem 2.6.2 and Theorem 2.3.7.
(ii): According to the Koszul formula (2.128), one has

$$2\Gamma^i_{jk} g_{il} = 2g(\nabla_{\partial_j}\partial_k, \partial_l) = \partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}, \tag{2.138}$$

where we have again used Theorem 2.3.7. Therefore,

$$\Gamma^i_{jk} = \Gamma^m_{jk} g_{ml} g^{li} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}). \tag{2.139}$$

(iii): Making use of the transformation rules Eqs. (2.34) and (2.35) as well as of the properties of the connection (cf. Def. 2.6.1), we find that

$$\begin{aligned}
\bar{\Gamma}^i_{jk}\bar{\partial}_i &= \nabla_{\bar{\partial}_j}\bar{\partial}_k = \nabla_{(\partial x^l/\partial \bar{x}^j)\partial_l}\left(\frac{\partial x^m}{\partial \bar{x}^k}\partial_m\right) = \frac{\partial x^l}{\partial \bar{x}^j}\nabla_{\partial_l}\left(\frac{\partial x^m}{\partial \bar{x}^k}\partial_m\right) && (2.140)\\
&= \frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x^m}{\partial \bar{x}^k}\nabla_{\partial_l}\partial_m + \frac{\partial x^l}{\partial \bar{x}^j}\partial_l\left(\frac{\partial x^m}{\partial \bar{x}^k}\right)\partial_m && (2.141)\\
&= \frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x^m}{\partial \bar{x}^k}\Gamma^i_{lm}\partial_i + \bar{\partial}_j\left(\frac{\partial x^l}{\partial \bar{x}^k}\right)\partial_l && (2.142)\\
&= \left(\frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x^m}{\partial \bar{x}^k}\frac{\partial \bar{x}^i}{\partial x^n}\Gamma^n_{lm} + \frac{\partial^2 x^l}{\partial \bar{x}^j\partial \bar{x}^k}\frac{\partial \bar{x}^i}{\partial x^l}\right)\bar{\partial}_i. && (2.143)
\end{aligned}$$

□

Some remarks:

- Equation (2.136) shows that the Christoffel symbols are not components of a tensor. The first term on the right-hand side is the 'tensor part' (cf. Theorem 2.4.5); however, the presence of the second term makes the Christoffel symbols not a tensor.

- For orthogonal coordinates, i.e., when the metric g_{ij} is of diagonal form, one obtains the simplified expressions

$$\begin{aligned}
\Gamma^i_{ii} &= \frac{1}{2} g^{ii}\partial_i g_{ii}, && (2.144)\\
\Gamma^j_{ii} &= -\frac{1}{2} g^{jj}\partial_j g_{ii}, \quad i \neq j, && (2.145)\\
\Gamma^j_{ij} &= \frac{1}{2} g^{jj}\partial_i g_{jj}, \quad i \neq j, && (2.146)\\
\Gamma^k_{ij} &= 0, \quad i, j, k \text{ pairwise different}, && (2.147)
\end{aligned}$$

and thus

$$\nabla_{\partial_i}\partial_i = \frac{1}{2}\left(g^{ii}\partial_i g_{ii}\partial_i - \sum_{k\neq i} g^{kk}\partial_k g_{ii}\partial_k\right) \tag{2.148}$$

$$\nabla_{\partial_i}\partial_j = \frac{1}{2}\left(g^{ii}\partial_j g_{ii}\partial_i - g^{jj}\partial_i g_{jj}\partial_j\right), \quad i \neq j. \tag{2.149}$$

Example. Let us calculate some useful explicit expressions for the Christoffel symbols of two-dimensional (pseudo-)Riemannian manifolds. Let us write the metric in the following form introduced by J. C. F. Gauss,

$$g_{ij} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \tag{2.150}$$

$$g^{ij} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}, \tag{2.151}$$

and let (u, v) denote the coordinates for a given chart ϕ. Furthermore, let subscripts 'u' or 'v' denote partial derivatives with respect to that coordinate. Because of the symmetry Eq. (2.134) there are six independent Christoffel symbols—from Eq. (2.135), we immediately find:

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FG_v}{2(EG - F^2)}, \quad \Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - F^2)}, \quad \Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, \tag{2.152}$$

$$\Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}, \quad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - F^2)}, \quad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}. \tag{2.153}$$

Exercise 2.7.4. *Consider the plane in polar coordinates $(u, v) = (r, \phi)$ as the two-dimensional Riemannian manifold $(\mathbb{R}^2, g)$ with metric*

$$\mathrm{d}s^2 = dr^2 + r^2\mathrm{d}\phi^2 \tag{2.154}$$

and compute the Christoffel symbols Γ^i_{jk} as well as the covariant derivatives $\nabla_{\partial_i}\partial_j$ of the coordinate fields.

Exercise 2.7.5. *Consider the sphere of radius r in $\mathbb{R}^3$ with spherical coordinates $(u, v) = (\Theta, \phi)$ as the two-dimensional Riemannian manifold (S^2, g) with metric*

$$\mathrm{d}s^2 = r^2\mathrm{d}\Theta^2 + r^2\sin^2\Theta\mathrm{d}\phi^2 \tag{2.155}$$

and compute the Christoffel symbols Γ^i_{jk} as well as the covariant derivatives $\nabla_{\partial_i}\partial_j$ of the coordinate fields.

Exercise 2.7.6. *Consider the Schwarzschild spacetime (which we will derive in Sec. 5.1) $M = \mathbb{R}\times(2m, \infty)\times S^2$ with coordinates (t, r, Θ, ϕ), where Θ and ϕ are the standard spherical coordinates of the unit sphere $S^2 \subset \mathbb{R}^3$, and metric*

$$g_{ij} = \begin{pmatrix} -h(r) & 0 & 0 & 0 \\ 0 & h(r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\Theta \end{pmatrix}. \tag{2.156}$$

Here, $h(r) = 1 - 2m/r$. Compute the Christoffel symbols and covariant derivatives $\nabla_{\partial_i}\partial_j$ of the coordinate vector fields.

Exercise 2.7.7. *Let ∇ be the Levi-Cività connection on M, and let $X, Y \in \mathcal{X}(M)$. Show the following useful properties related to the Christoffel symbols associated with ∇:*

(i) 'Contraction' of Christoffel symbols:

$$\Gamma^i_{ij} = \frac{1}{\sqrt{|g|}}\partial_j\sqrt{|g|}, \quad \text{where } g = \det(g) \tag{2.157}$$

(ii) Divergence of a vector field:

$$\nabla_i X^i = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}X^i\right) \tag{2.158}$$

(iii) Verify by explicit computation in local coordinates that for any torsion-free connection ∇, the Lie bracket can be written in terms of covariant derivatives:

$$[X, Y]^i = X^j\nabla_j Y^i - Y^j\nabla_j X^i, \quad \text{where } \nabla_i = \nabla_{\partial_i}. \tag{2.159}$$

Note: if ∇ is the Levi-Cività connection, this identity holds by construction (cf. the Fundamental Theorem of Riemannian geometry 2.6.2).

2.8 Parallel transport

Let (M, g) be an n-dimensional (pseudo-)Riemannian manifold with Levi-Civita connection ∇.

Definition 2.8.1. *Let $c : I \subset \mathbb{R} \to M$ be a curve on M with tangent field $\dot{c}(t)$. A vector field $X \in \mathcal{X}(M)$ is said to be* ***parallel*** *or* ***parallel transported*** *along c if*

$$\frac{\mathrm{D}X}{\mathrm{d}t} \equiv \nabla_{\dot{c}(t)}X = 0 \tag{2.160}$$

for all t. The vector $\frac{\mathrm{D}X}{\mathrm{d}t}$ is called ***covariant derivative along*** *c.*

Writing in terms of coordinate components on a given chart (U, ϕ), $X = X^i\partial_i$, $\dot{c}(t) = \frac{\mathrm{d}x^i}{\mathrm{d}t}\partial_i$, one can express the covariant derivative along c as

$$\nabla_{\dot{c}(t)}X = \left(\frac{\mathrm{d}X^i}{\mathrm{d}t} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}X^k\right)\partial_i, \tag{2.161}$$

where we have adopted the notation $\mathrm{d}X^i/\mathrm{d}t \equiv (X^i \circ c)'(t) = [\dot{c}(t)]^j\partial_j X^i = \frac{\mathrm{d}x^j}{\mathrm{d}t}\partial_j X^i$ and $\frac{\mathrm{d}x^j}{\mathrm{d}t} = (\phi^j \circ c)'(t)$, with ϕ^j being the projection onto the j-th coordinate. The condition for a parallel transported vector field thus reads in components:

$$\frac{\mathrm{d}X^i}{\mathrm{d}t} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}X^k = 0, \quad i = 1, \ldots, n. \tag{2.162}$$

Note that Eq. (2.162) is a system of homogeneous linear differential equations for the real functions $X^i \circ \phi$. Therefore, given initial values $X^i(c(t_0))$ and $(\mathrm{d}X^i/\mathrm{d}t)(t_0) = (X^i \circ c)'(t_0)$ for some $t_0 \in I$ there exists a unique solution. Hence, for vector fields $X, Y \in \mathcal{X}(M)$ that are both parallel along c and $X(c(t_0)) = Y(c(t_0))$ for some t_0, it follows that $X(c(t)) = Y(c(t))$ for all t. This means that every vector in $T_{c(t_1)}M$ can be uniquely mapped to a vector in $T_{c(t_2)}M$ by parallel transport along c from $c(t_1)$ to $c(t_2)$, i.e., parallel transport creates an isomorphism between the tangent spaces. This leads to the following

Definition 2.8.2. *Let $c : I \subset \mathbb{R} \to M$ be a curve on M and $t_1, t \in I$. The linear isomorphism*

$$\tau_{t,t_1} : T_{c(t_1)}M \to T_{c(t)}M, \tag{2.163}$$

*which maps a vector $v \in T_{c(t_1)}M$ to the parallel transported vector $v(t) \in T_{c(t)}M$ along c, is called **parallel transport along** c from $c(t_1)$ to $c(t)$.*

Some remarks.

- Because of linearity and uniqueness of Eqs. (2.162), τ_{t,t_1} is linear and injective. Since the tangent spaces have the same dimensions, $\tau_{t_1,t}$ is also surjective, hence bijective.
- By means of parallel transport, the connection ∇ thus 'connects' tangent spaces at different points on the manifold, hence the name **'connection'**.
- Due to uniqueness of Eqs. (2.162), one has

$$\tau_{t_2,s} \circ \tau_{s,t_1} = \tau_{t_2,t_1}, \quad \tau_{t_1,t_1} = \mathrm{Id}, \quad \tau_{t_2,t_1}^{-1} = \tau_{t_1,t_2}. \tag{2.164}$$

- Note that from Eq. (2.161) we immediately obtain an expression for the change of the coordinate components of a vector field X under parallel transport along a curve c:

$$\mathrm{d}X^i = -\Gamma^i_{jk} X^j \mathrm{d}x^k. \tag{2.165}$$

  For a finite path length, we obtain the component expression for parallel transport:

$$[\tau_{t,s}X]^i = (\tau_{t,s})^i_j X^j = \int_s^t -\Gamma^i_{jk} X^j \mathrm{d}x^k, \tag{2.166}$$

  where, again, $\mathrm{d}x^k$ is the k-th component of the line element along c.

Parallel transport owes its name to the fact that the length of vectors and of the angles between them remain constant under this operation, which we will show now.

Theorem 2.8.3. *Let $c : I \subset \mathbb{R} \to M$ be a curve on M and $v_1, v_2 \in T_{c(t_0)}M$ for $t_0 \in I$. Then for all $t \in I$, $g(\tau_{t,t_0}v_1, \tau_{t,t_0}v_2) = g(v_1, v_2)$.*

Proof. Let $X, Y \in \mathcal{X}(M)$ be two parallel vector fields along c with $X(c(t_0)) = v_1$ and $Y(c(t_0)) = v_2$. Such vector fields exist due to existence and uniqueness of solutions to Eq. (2.162). Then we find using the Ricci identity from Theorem 2.6.2:

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\tau_{t,t_0}v_1, \tau_{t,t_0}v_2) = \frac{\mathrm{d}}{\mathrm{d}t} g(X_{c(t)}, Y_{c(t)}) = \dot{c}(t) g(X_{c(t)}, Y_{c(t)}) \tag{2.167}$$

$$= g(\nabla_{\dot{c}(t)}X, Y) + g(X, \nabla_{\dot{c}(t)}Y) = g(0, Y) + g(X, 0) = 0. \tag{2.168}$$

□

It is important to note that parallel transport is *not* independent of the curve used: parallel transport between the same start and end points along different curves on a manifold leads, in general, to different parallel transported vectors, i.e., parallel transport depends on the path taken. Consider, for example, parallel transport of a vector along a closed loop on a sphere including the north pole, following constant lines of longitude and latitude. Starting at the equator and first transporting along the equator and then along a line of constant longitude toward the pole, $\tau_{t_{\text{pole}},t_{\text{eq},2}} \circ \tau_{t_{\text{eq},2},t_{\text{eq},1}}$, results in a different vector than directly transporting along constant longitude to the pole $\tau_{t_{\text{pole}},t_{\text{eq},1}}$. In other words, transporting a vector along a closed path back to the equator, $\tau_{t_{\text{eq},1},t_{\text{pole}}} \circ \tau_{t_{\text{pole}},t_{\text{eq},2}} \circ \tau_{t_{\text{eq},2},t_{\text{eq},1}}$ effectively results in a rotation of the vector. This is at the heart of the **Gauss-Bonnet theorem** in differential geometry, which states that the angle of rotation of a parallel transported tangent vector along a simple closed curve on a two-dimensional manifold equals the integral of the Gaussian curvature over the enclosed area. Rotation of a tangent vector along a simple closed curve is thus an indicator of curvature.

As parallel transport provides a means to connect vectors from different tangent spaces of the manifold, it also allows for a reinterpretation of the covariant derivative in a way very similar to the definition of the directional derivative of vector fields in flat space (Eq. (2.121)), as we will now show.

Theorem 2.8.4. *Let $c : I \subset \mathbb{R} \to M$ be a curve on M and $X \in \mathcal{X}(M)$ be a vector field along c. The covariant derivative along c can then be expressed as*

$$\nabla_{\dot{c}(t)} X = \lim_{h\to 0} \frac{1}{h} \left(\tau_{t,t+h} X_{c(t+h)} - X_{c(t)} \right) \tag{2.169}$$

$$= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \tau_{t,s} X_{c(s)} \tag{2.170}$$

$$\equiv \dot{v}(t), \tag{2.171}$$

where we defined $v(s) = \tau_{t,s} X_{c(s)}$.

Proof. Since $v = v(s)$ is the parallel transported vector $v_0 = X_{c(t)} \in T_{c(t)}M$, v satisfies Eq. (2.162),

$$\dot{v}^i + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}s} v^k = 0. \tag{2.172}$$

In components, we can write

$$X^i_{c(s)} = (\tau_{s,t})^i_j v^j. \tag{2.173}$$

This yields

$$(X^i \circ c)'(s) = \frac{\partial}{\partial s} \left[(\tau_{s,t})^i_j \right] v^j + (\tau_{s,t})^i_j \dot{v}^j(s). \tag{2.174}$$

Writing Eq. (2.172) in terms of $v(s) = \tau_{s,t} v_0$, we find

$$\frac{\partial}{\partial s} (\tau_{s,t})^i_j = -\Gamma^i_{lk} \frac{\mathrm{d}x^l}{\mathrm{d}s} (\tau_{s,t})^k_j. \tag{2.175}$$

Therefore, one may write:

$$\frac{\mathrm{d}X^i}{\mathrm{d}s} = (X^i \circ c)'(s) = -\Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}s} (\tau_{s,t})^k_j v^j + (\tau_{s,t})^i_j \dot{v}^j(s). \tag{2.176}$$

Evaluating at $s = t$ we find that

$$\dot{v}^i = \frac{\mathrm{d}X^i}{\mathrm{d}t} + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}s} X^k_{c(t)}, \tag{2.177}$$

which is the i-th component of $\nabla_{\dot{c}(t)} X$ (cf. Eq. (2.161)). □

We can now generalize the concept of parallel transport to covector and tensor fields.

Parallel transport of covector and tensor fields. Let us start by defining parallel transport for covector fields:

Definition 2.8.5. *Let $c : I \subset \mathbb{R} \to M$ be a curve on M, and $\alpha \in T^*_{c(s)}M$ a covector. The parallel transported covector $\tau_{t,s}\alpha \in T^*_{c(t)}M$ is defined by*

$$\tau_{t,s}\alpha(v) = \alpha(\tau_{s,t}v) \quad \textit{for all } v \in T_{c(t)}M. \tag{2.178}$$

Some remarks.

- Note that, due to the identities (2.164), Eq. (2.178) is equivalent to requiring that

$$\tau_{t,s}\alpha(\tau_{t,s}v) = \alpha(v), \quad \text{for all } v \in T_{c(s)}M. \tag{2.179}$$

  Indeed, Eq. (2.178) is often written in the following way

$$\langle \tau_{t,s}\alpha, \tau_{t,s}v \rangle = \langle \alpha, v \rangle, \tag{2.180}$$

  where $\langle \cdot \rangle$ is the inner product on the tangent spaces induced by the metric. In components, we also have

$$(\tau_{t,s}\alpha)_i(\tau_{t,s}v)^i = \alpha_i v^i. \tag{2.181}$$

- Equation (2.178) shows that parallel transport of covectors $\tau^*_{r,s} : T^*_{c(s)}M \to T^*_{c(t)}M$ is the corresponding dual map to $\tau_{r,s} : T_{c(s)}M \to T_{c(t)}M$. Usually, as in already in Def. 2.8.5, we do not make any distinction in writing and denote $\tau^*_{r,s}$ by $\tau_{r,s}$.

With this definition, we can now generalize to tensor fields:

Definition 2.8.6. *Let $c : I \subset \mathbb{R} \to M$ be a curve on M, and $T \in T_{c(s)}M^p_q$ a (p,q)-tensor. The parallel transported tensor $\tau_{t,s}T \in T_{c(t)}M^p_q$ is defined by*

$$\tau_{t,s}T(\alpha^1, \ldots, \alpha^p, v_1, \ldots, v_q) = T(\tau_{s,t}(\alpha^1), \ldots, \tau_{s,t}(\alpha^p), \tau_{s,t}(v_1), \ldots, \tau_{s,t}(v_q)) \tag{2.182}$$

for all $\alpha^1, \ldots, \alpha^p \in T^*_{c(t)}M$ and $v_1, \ldots, v_q \in T_{c(t)}M$.

Again, Eq. (2.182) is equivalent to requiring

$$\tau_{t,s}T(\tau_{t,s}\alpha^1, \ldots, \tau_{t,s}\alpha^p, \tau_{t,s}v_1, \ldots, \tau_{t,s}v_q) = T(\alpha^1, \ldots, \alpha^p, v_1, \ldots, v_q) \tag{2.183}$$

for all $\alpha^1, \ldots, \alpha^p \in T^*_{c(s)}M$ and $v_1, \ldots, v_q \in T_{c(s)}M$.

The following statement that parallel transport commutes with several tensor operations is of fundamental importance to practical calculations in general relativity. This gives rise to another theorem stating that the covariant derivative commutes with these tensor operations (see next section, Theorem 2.9.3).

Theorem 2.8.7. *Parallel transport commutes with raising and lowering indices as well as with contraction of tensors. This means that for a curve $c : I \subset \mathbb{R} \to M$ on M, $l \in \{1, \ldots, r\}$, $m \in \{1, \ldots, s\}$, and a tensor $T \in T_{c(s)}M^p_q$, we have*

$$\begin{aligned}
\tau_{t,s}(T^{\flat l}) &= g^{\flat l}(\tau_{t,s}T), && (2.184)\\
\tau_{t,s}(T^{\sharp m}) &= g^{\sharp m}(\tau_{t,s}T), && (2.185)\\
\tau_{t,s}(C^l_m T) &= C^l_m(\tau_{t,s}T). && (2.186)
\end{aligned}$$

Proof. Let $w_1, \ldots, w_{q+1} \in T_{c(t)}M$ and $\beta_1, \ldots, \beta_{p+1} \in T^*_{c(t)}M$. Using the definition of parallel transported tensors given above, we can then explicitly compute:

$$\begin{aligned}
&g^{\sharp m}(\tau_{t,s}T)(\beta^1, \ldots, \beta^{m-1}, w_{q+1}, \beta^{m+1}, \ldots, \beta^p, w_1, \ldots, w_q) && (2.187)\\
&= \tau_{t,s}T(\beta^1, \ldots, \beta^{m-1}, (w_{q+1})^\flat, \beta^{m+1}, \ldots, \beta^p, w_1, \ldots, w_q) && (2.188)\\
&= T(\tau_{s,t}\beta^1, \ldots, \tau_{s,t}\beta^{m-1}, \tau_{s,t}(w_{q+1})^\flat, \tau_{s,t}\beta^{m+1}, \ldots, \tau_{s,t}\beta^p, \tau_{s,t}w_1, \ldots, \tau_{s,t}w_q) && (2.189)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\tau_{t,s}(T^{\sharp m})(\beta^1, \ldots, \beta^{m-1}, w_{q+1}, \beta^{m+1}, \ldots, \beta^p, w_1, \ldots, w_q) && (2.190)\\
&= T^{\sharp m}(\tau_{s,t}\beta^1, \ldots, \tau_{s,t}\beta^{m-1}, \tau_{s,t}w_{q+1}, \tau_{s,t}\beta^{m+1}, \ldots, \tau_{s,t}\beta^p, \tau_{s,t}w_1, \ldots, \tau_{s,t}w_q) && (2.191)\\
&= T(\tau_{s,t}\beta^1, \ldots, \tau_{s,t}\beta^{m-1}, (\tau_{s,t}w_{q+1})^\flat, \tau_{s,t}\beta^{m+1}, \ldots, \tau_{s,t}\beta^p, \tau_{s,t}w_1, \ldots, \tau_{s,t}w_q) && (2.192)
\end{aligned}$$

According to Def. 2.8.5 and Theorem 2.8.3, one has for $v \in T_{c(s)}M$, $w \in T^*_{c(t)}M$:

$$\begin{aligned}
(\tau_{s,t}w^\flat)(v) &= w^\flat(\tau_{t,s}v) = g(w, \tau_{t,s}v) && (2.193)\\
&= g(\tau_{s,t}w, v) = (\tau_{s,t}w)^\flat(v), && (2.194)
\end{aligned}$$

which shows that Eq. (2.189) and (2.192) are identical, i.e., that Eq. (2.185) holds. Therefore, raising indices commutes with parallel transport, $g^{\sharp m} \circ \tau_{t,s} = \tau_{t,s} \circ g^{\sharp m}$. Taking the inverse of this shows that Eq. (2.184) holds:

$$\begin{aligned}
g^{\flat l} \circ \tau_{t,s} &= (g^{\sharp l})^{-1} \circ \tau_{s,t}^{-1} = (\tau_{s,t} \circ g^{\sharp l})^{-1} && (2.195)\\
&= (g^{\sharp l} \circ \tau_{s,t})^{-1} = \tau_{s,t}^{-1} \circ (g^{\sharp l})^{-1} && (2.196)\\
&= \tau_{t,s} \circ g^{\flat l}. && (2.197)
\end{aligned}$$

Similarly, for contractions we compute explicitly:

$$\begin{aligned}
&C^l_m(\tau_{t,s}T)(\beta^1, \ldots, \beta^{l-1}, \beta^{l+1}, \ldots, \beta^p, w_1, \ldots, w_{l-1}, w_{l+1}, \ldots, w_q) && (2.198)\\
&= \tau_{t,s}T(\beta^1, \ldots, \beta^{l-1}, \tau_{t,s}\alpha^k, \beta^{l+1}, \ldots, \beta^p, w_1, \ldots, w_{l-1}, \tau_{t,s}v_k, w_{l+1}, \ldots, w_q) && (2.199)\\
&= T(\tau_{s,t}\beta^1, \ldots, \tau_{s,t}\beta^{l-1}, \alpha^k, \tau_{s,t}\beta^{l+1}, \ldots, \tau_{s,t}\beta^p, && (2.200)\\
&\qquad \tau_{s,t}w_1, \ldots, \tau_{s,t}w_{l-1}, v_k, \tau_{s,t}w_{l+1}, \ldots, \tau_{s,t}w_q), && (2.201)
\end{aligned}$$

where $\{v_i\}$ is a basis of $T_{c(s)}M$ and $\{\alpha_i\}$ is the corresponding basis of $T^*_{c(s)}M$. However, we also have:

$$\begin{aligned}
&\tau_{t,s}(C^l_m T)(\beta^1, \ldots, \beta^{l-1}, \beta^{l+1}, \ldots, \beta^p, w_1, \ldots, w_{l-1}, w_{l+1}, \ldots, w_q) && (2.202)\\
&= C^l_m T(\tau_{s,t}\beta^1, \ldots, \tau_{s,t}\beta^{l-1}, \tau_{s,t}\beta^{l+1}, \ldots, \tau_{s,t}\beta^p, && (2.203)\\
&\qquad \tau_{s,t}w_1, \ldots, \tau_{s,t}w_{l-1}, \tau_{s,t}w_{l+1}, \ldots, \tau_{s,t}w_q) && (2.204)\\
&= T(\tau_{s,t}\beta^1, \ldots, \tau_{s,t}\beta^{l-1}, \alpha^k, \tau_{s,t}\beta^{l+1}, \ldots, \tau_{s,t}\beta^p, && (2.205)\\
&\qquad \tau_{s,t}w_1, \ldots, \tau_{s,t}w_{l-1}, v_k, \tau_{s,t}w_{l+1}, \ldots, \tau_{s,t}w_q). && (2.206)
\end{aligned}$$

□

Exercise 2.8.8. *Consider the manifold $M = \mathbb{R}^2$ with polar coordinates $(x_1, x_2) = (r, \phi)$. Consider the vector $x = e_r$ at $(r, \phi) = (1, 0)$, where e_r is the unit vector in $r-$direction, and compute its parallel transported vector at $(r, \phi) = (1, \pi/2)$ along the unit circle.*

Exercise 2.8.9. *Consider the manifold $M = S^2$, i.e., the unit sphere, with spherical polar coordinates $(x_1, x_2) = (\Theta, \phi)$. Consider the vector $x = e_\phi$ at $(\Theta, \phi) = (\pi/2, 0)$ on the equator, where e_ϕ is the unit vector in ϕ-direction, and compute its parallel transported vector at the same location along the closed path*

$$(\Theta, \phi) = (\pi/2, 0) \to (\epsilon, 0) \to (\epsilon, \pi/2) \to (\pi/2, \pi/2) \to (\pi/2, 0), \tag{2.207}$$

where $\to$ means following paths of constant longitude or latitude. Here, ϵ is a small but finite number, introduced to avoid the coordinate singularity at $\Theta = 0$. By how much does the angle of the vector change parallel transporting it once around the closed loop?

2.9 Covariant derivative of tensor fields

Let (M, g) be an n-dimensional (pseudo-)Riemannian manifold with Levi-Civita connection ∇.

Using the concept of parallel transport of tensor fields (see Sec. 2.8, Def. 2.8.6), one can generalize the covariant derivative of vector fields (see Sec. 2.6) to tensor fields. The starting point for this is the form of the covariant derivative along a curve obtained in Theorem 2.8.4, which can be generalized to tensor fields in a straightforward manner:

Definition 2.9.1. *Let $c : I \subset \mathbb{R} \to M$ be a curve on M, and $T \in \mathcal{T}^r_s(M)$ a (r,s)-tensor field along c. The* ***covariant derivative of*** T ***along*** c *is defined by*

$$\begin{aligned} \nabla_{\dot c(t)} T &= \lim_{h \to 0} \frac{1}{h} \left(\tau_{t,t+h} T_{c(t+h)} - T_{c(t)} \right) & (2.208) \\ &= \left. \frac{\mathrm{d}}{\mathrm{d}s} \right|_{s=t} \tau_{t,s} T_{c(s)}. & (2.209) \end{aligned}$$

The ***covariant derivative*** $\nabla_X T \in \mathcal{T}^r_s(M)$ ***of*** T ***in direction of a vector field*** $X \in \mathcal{X}(M)$ *is then defined pointwise by*

$$(\nabla_X T)_p = \nabla_{X_p} T = \nabla_{\dot c(t)} T \tag{2.210}$$

for $p \in M$, where c is a curve with $c(t) = p$ and $\dot c(t) = X_p$.[1] *For $f \in \mathcal{F}(M) \simeq \mathcal{T}^0_0(M)$, we set $\nabla_X f = X(f)$.*

We will now derive a few rules for computing covariant derivatives of tensor fields that prove useful in practical calculations. They will also allow us to derive expressions for the corresponding coordinate components of the covariant derivatives of tensors.

Theorem 2.9.2. ***(Product rule for tensors).*** *Let $X \in \mathcal{X}(M)$ a vector field on M and $S \in \mathcal{T}^r_s(M)$, $T \in \mathcal{T}^p_q(M)$ tensor fields of rank (r, s) and (p, q), respectively. Then*

$$\nabla_X (S \otimes T) = \nabla_X S \otimes T + S \otimes \nabla_X T. \tag{2.211}$$

[1] Note that such a vector field exists due to existence (and uniqueness) of solutions to Eq. (2.162).

Proof. Let $p \in M$ and $c : I \subset \mathbb{R} \to M$ be a curve with $\dot{c}(0) = X_p$. Then:

$$\begin{aligned}
\nabla_{\dot{c}(0)}(S \otimes T) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left[\tau_{0,s}(S_{c(s)} \otimes T_{c(s)})\right] && (2.212)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left[\tau_{0,s}S_{c(s)} \otimes \tau_{0,s}T_{c(s)}\right] && (2.213)\\
&= \left[\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \tau_{0,s}S_{c(s)}\right] \otimes \tau_{0,0}T_{c(0)} + \tau_{0,0}S_{c(0)} \otimes \left[\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \tau_{0,s}T_{c(s)}\right] && (2.214)\\
&= \nabla_{\dot{c}(0)}S \otimes T + S \otimes \nabla_{\dot{c}(0)}T. && (2.215)
\end{aligned}$$

□

Theorem 2.9.3. *The covariant derivative of a tensor along a vector field commutes with raising and lowering indices, as well as with contractions. In formulae, let $X \in \mathcal{X}(M)$ be a vector field on M, $T \in \mathcal{T}^r_s(M)$ a tensor field of rank (r, s), $l \in \{1, \dots, r\}$, $m \in \{1, \dots, s\}$. Then:*

$$\begin{aligned}
\nabla_X(T^{\flat l}) &= g^{\flat l}(\nabla_X T), && (2.216)\\
\nabla_X(T^{\sharp m}) &= g^{\sharp m}(\nabla_X T), && (2.217)\\
\nabla_X(C^l_m T) &= C^l_m(\nabla_X T). && (2.218)
\end{aligned}$$

Proof. This theorem largely follows from Theorem 2.8.7. Let $p \in M$ and $c : I \subset \mathbb{R} \to M$ be a curve with $\dot{c}(0) = X_p$. Then for any $p \in M$:

$$\nabla_{X_p}(T^{\flat l}) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \tau_{0,s}T^{\flat l}_{c(s)} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} g^{\flat l}(\tau_{0,s}T_{c(s)}) = g^{\flat l}(\nabla_{X_p}T). \quad (2.219)$$

The remaining identities are obtained analogously. □

Theorem 2.9.4. *Let $T \in \mathcal{T}^r_s(M)$ be a (r,s)-tensor field, $X, Y_1, \dots, Y_s \in \mathcal{X}(M)$ vector fields, and $\omega^1, \dots, \omega^r \in \mathcal{X}^*(M)$ covector fields on M. The covariant derivative of T with respect to X can then be written as*

$$\begin{aligned}
&\nabla_X T(\omega^1, \dots, \omega^r, Y_1, \dots, Y_s) =\\
&\qquad X(T(\omega^1, \dots, Y_s)) - T(\nabla_X\omega^1, \omega^2, \dots, Y_s) - \dots - T(\omega^1, \dots, Y_{s-1}, \nabla_X Y_s). \quad (2.220)
\end{aligned}$$

Proof. Consider the vector product of T with the vector fields and covector fields $T \otimes Y_1 \otimes \dots \otimes Y_s \otimes \omega^1 \otimes \dots \otimes \omega^r$. According to Theorem 2.9.2 one has the identity:

$$\begin{aligned}
&\nabla_X(T \otimes Y_1 \otimes \dots \otimes Y_s \otimes \omega^1 \otimes \dots \otimes \omega^r) =\\
&\qquad \nabla_X T \otimes Y_1 \otimes \dots \otimes \omega^r + T \otimes \nabla_X Y_1 \otimes \dots \otimes \omega^r + \dots + T \otimes Y_1 \otimes \dots \otimes \nabla_X\omega^r. \quad (2.221)
\end{aligned}$$

Let $\partial_1, \dots, \partial_n$ and $\mathrm{d}x^1, \dots, \mathrm{d}x^n$ denote the basis vector and covector fields in a chart (U, ϕ). Observe that the total contraction of a tensor of the aforementioned type is given by

$$\begin{aligned}
&T \otimes Y_1 \otimes \dots \otimes Y_s \otimes \omega^1 \otimes \dots \otimes \omega^r(\mathrm{d}x^{i_1}, \dots, \mathrm{d}x^{i_{r+s}}, \partial_{i_1}, \dots, \partial_{i_{r+s}}) && (2.222)\\
&= T(\mathrm{d}x^{i_1}, \dots, \mathrm{d}x^{i_r}, \partial_{i_{r+1}}, \dots, \partial_{i_{r+s}})\mathrm{d}x^{i_{r+1}}(Y_1) \dots \mathrm{d}x^{i_{r+s}}(Y_s)\omega^1(\partial_{i_1}) \dots \omega^r(\partial_{i_r}) && (2.223)\\
&= T(\omega^1(\partial_{i_1})\mathrm{d}x^{i_1}, \dots, \omega^r(\partial_{i_r})\mathrm{d}x^{i_r}, \mathrm{d}x^{i_{r+1}}(Y_1)\partial_{i_{r+1}}, \dots, \mathrm{d}x^{i_{r+s}}(Y_s)\partial_{i_{r+s}}) && (2.224)\\
&= T(\omega^1, \dots, \omega^r, Y_1, \dots, Y_s), && (2.225)
\end{aligned}$$

i.e., taking the total contraction of such a tensor product simply amounts to evaluating the original tensor T on the additional vector fields $Y_1, \ldots, Y_s$ and covector fields $\omega^1, \ldots, \omega^r$. Taking the total contraction of Eq. (2.221) and using Theorem 2.9.3, one finds:

$$\nabla_X(T(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s)) = \\ (\nabla_X T)(\omega^1, \ldots, Y_s) + T(\nabla_X \omega^1, \ldots, Y_s) + \ldots + T(\omega^1, \ldots, \nabla_X Y_s), \quad (2.226)$$

and, therefore, using $\nabla_X f = X(f)$ (Def. 2.9.1),

$$(\nabla_X T)(\omega^1, \ldots, Y_s) = \\ X(T(\omega^1, \ldots, Y_s)) - T(\nabla_X \omega^1, \ldots, Y_s) - \ldots - T(\omega^1, \ldots, \nabla_X Y_s). \quad (2.227)$$

□

Note that according to Def. 2.9.1, the covariant derivative of a tensor in direction of X only depends on the vector field locally, i.e., at $p \in M$ it only depends on X_p. The identity in Theorem 2.9.4 shows that it does so linearly. One can therefore reinterpret the covariant derivative of a tensor field as a linear mapping $\mathcal{T}^r_s(M) \to \mathcal{T}^r_{s+1}(M)$ in the following sense:

Definition 2.9.5. *The (multi-)linear mapping*

$$\nabla : \mathcal{T}^r_s(M) \rightarrow \mathcal{T}^r_{s+1}(M) \quad (2.228)$$
$$T \mapsto \nabla T \quad (2.229)$$

defined by

$$(\nabla T)(\omega^1, \ldots, \omega^r, X, Y_1, \ldots, Y_s) = \nabla_X T(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s), \quad (2.230)$$

where $\omega^1, \ldots, \omega^r \in \mathcal{X}^*(M)$ and $X, Y_1, \ldots, Y_s \in \mathcal{X}(M)$, is called ***covariant derivative*** *of a tensor field T.*

Some remarks.

- Taking the covariant derivative of a tensor T increases the covariant rank of T by $+1$, hence the name **'covariant'** derivative.

- Note that the covariant derivative of tensors in the special cases of vector fields $X \in \mathcal{T}^0_1$ reduces to the corresponding definition of the covariant derivative of a vector field defined earlier in Sec. 2.6.

- From the above theorem and the Ricci identity (cf. the fundamental theorem of Riemannian geometry 2.6.2), it immediately follows that the covariant derivative of the metric tensor along any vector field vanishes, $\nabla_X g \equiv 0$. This condition is sometimes referred to as **metric compatibility**, and it is, in fact, equivalent to the Ricci identity; this is the subject of Exercise 2.9.7.

Component expressions in local coordinates. Let us recall the local expansion of a tensor field $T \in \mathcal{T}^r_s(M)$ in local coordinate tensor basis fields from Sec. 2.4,

$$T = T^{i_1 \dots i_r}_{j_1 \dots j_s} \partial_{i_1} \otimes \dots \otimes \partial_{i_r} \otimes \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_s}. \tag{2.231}$$

Let us also introduce the notations

$$(\nabla_{\partial_k} T)^{i_1 \dots i_r}_{j_1 \dots j_s} = (\nabla_k T)^{i_1 \dots i_r}_{j_1 \dots j_s} = T^{i_1 \dots i_r}_{j_1 \dots j_s;k}, \tag{2.232}$$

$$\partial_k T^{i_1 \dots i_r}_{j_1 \dots j_s} = T^{i_1 \dots i_r}_{j_1 \dots j_s,k} \tag{2.233}$$

According to Def. (2.2.9), we can then also write

$$X(T^{i_1 \dots i_r}_{j_1 \dots j_s}) = X^k T^{i_1 \dots i_r}_{j_1 \dots j_s,k} \tag{2.234}$$

for $X = X^k \partial_k \in \mathcal{X}(M)$.

Theorem 2.9.6. *In a chart (U, ϕ), the components of the covariant derivative of a tensor field $T \in \mathcal{T}^r_s(M)$ are given by*

$$(\nabla T)^{i_1 \dots i_r}_{kj_1 \dots j_s} = T^{i_1 \dots i_r}_{j_1 \dots j_s;k} = \\ T^{i_1 \dots i_r}_{j_1 \dots j_s,k} + \Gamma^{i_1}_{kl_1} T^{l_1 i_2 \dots i_r}_{j_1 \dots j_s} + \dots + \Gamma^{i_r}_{kl_r} T^{i_1 \dots i_{r-1} l_r}_{j_1 \dots j_s} - \Gamma^{m_1}_{kj_1} T^{i_1 \dots i_r}_{m_1 j_2 \dots j_s} - \Gamma^{m_s}_{kj_s} T^{i_1 \dots i_r}_{j_1 \dots j_{s-1} m_s}. \tag{2.235}$$

Proof. The components of the covariant derivative are defined as (cf. Eq. (2.86))

$$(\nabla T)^{i_1 \dots i_r}_{kj_1 \dots j_s} = \nabla T(\mathrm{d}x^{i_1}, \dots, \mathrm{d}x^{i_r}, \partial_k, \partial_{j_1}, \dots, \partial_{j_s}). \tag{2.236}$$

We will now explicitly evaluate the formula in Theorem 2.9.4 term by term. The first term yields:

$$\partial_k (T(\mathrm{d}x^{i_1}, \dots, \mathrm{d}x^{i_r}, \partial_{j_1}, \dots, \partial_{j_s})) = T^{i_1 \dots i_r}_{j_1 \dots j_s,k}. \tag{2.237}$$

Terms involving covariant derivatives of basis vector fields give rise to terms of the form

$$- T(\mathrm{d}x^{i_1}, \dots, \mathrm{d}x^{i_r}, \partial_{j_1}, \dots, \nabla_{\partial_k} \partial_{j_l}, \dots, \partial_{j_s}) = -\Gamma^m_{kj_l} T^{i_1 \dots i_r}_{j_1 \dots m \dots j_s} \tag{2.238}$$

(cf. the definition of the Christoffel symbols, Def. 2.7.1). In order to compute terms involving covariant derivatives of basis covector fields, we start by evaluating the identity from Theorem 2.9.2 using a vector field Y and covector field ω:

$$\nabla_X (Y \otimes \omega) = \nabla_X Y \otimes \omega + Y \otimes \nabla_X \omega. \tag{2.239}$$

After total contraction as in the proof of Theorem 2.9.4 and using Theorem 2.9.3, one obtains:

$$\nabla_X (\omega(Y)) = \omega(\nabla_X Y) + (\nabla_X \omega)(Y). \tag{2.240}$$

Specifically, using basis fields,

$$(\nabla_{\partial_k} \mathrm{d}x^i)(\partial_j) = \nabla_{\partial_k} (\mathrm{d}x^i(\partial_j)) - \mathrm{d}x^i(\nabla_{\partial_k} \partial_j) = -\Gamma^i_{jk}. \tag{2.241}$$

Therefore, terms involving covariant derivatives of basis covector fields lead to expressions of the form

$$- T(\mathrm{d}x^{i_1}, \dots, \nabla_{\partial_k} \mathrm{d}x^{i_l}, \dots, \mathrm{d}x^{i_r}, \partial_{j_1}, \dots, \partial_{j_s}) = \Gamma^{i_l}_{kn} T^{i_1 \dots n \dots i_r}_{j_1 \dots j_s}. \tag{2.242}$$

□

Example. Let us explicitly evaluate Theorem 2.9.6 for vector, covector and rank-2 tensor fields:

$$X^i{}_{;j} = X^i{}_{,j} + \Gamma^i_{jk} X^k, \tag{2.243}$$

$$\omega_{i;j} = \omega_{i,j} - \Gamma^k_{ij} \omega_k, \tag{2.244}$$

$$T^{ij}{}_{;k} = T^{ij}{}_{,k} + \Gamma^i_{kl} T^{lj} + \Gamma^j_{kl} T^{il}, \tag{2.245}$$

$$T^i{}_{j;k} = T^i{}_{j,k} + \Gamma^i_{kl} T^l{}_j - \Gamma^l_{kj} T^i{}_l. \tag{2.246}$$

$$T_{ij;k} = T_{ij,k} - \Gamma^l_{ki} T_{lj} - \Gamma^l_{kj} T_{il}. \tag{2.247}$$

Furthermore, we note that as a consequence of the identity (2.157), the components of the divergence of rank-2 tensor fields can be written as

$$T^{ij}{}_{;j} = T^{ij}{}_{,j} + \Gamma^i_{jl} T^{lj} + \Gamma^j_{jl} T^{il} \tag{2.248}$$

$$= \frac{1}{\sqrt{|g|}} \partial_j (\sqrt{|g|} T^{ij}) + \Gamma^i_{jl} T^{lj}. \tag{2.249}$$

Exercise 2.9.7. *Show that the covariant derivative of the metric tensor $g \in \mathcal{T}^0_2(M)$ vanishes, $\nabla_X g = 0$ for any vector field $X \in \mathcal{X}(M)$. Show that this "**metric compatibility condition**" is equivalent to the Ricci identiy (cf. fundamental theorem of Riemannian geometry 2.6.2).*

Exercise 2.9.8. *Consider the unit sphere $M = S^2$ from Exercise 2.8.9 with polar coordinates $(x^1, x^2) = (\theta, \phi)$. Recall that the only non-zero Christoffel symbols are*

$$\Gamma^1_{22} = -\sin\theta\cos\theta, \quad \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta. \tag{2.250}$$

Calculate the covariant derivative $\nabla_\mu X^\nu$ of the vector field $X^\mu = (\sin\phi, \cot\theta\cos\phi)$. Compute the components of the corresponding covector field X_μ and find $\nabla_\mu X_\nu$. Use these results to show that

$$\nabla_\mu X_\nu = g_{\nu\gamma} \nabla_\mu X^\gamma. \tag{2.251}$$

This should not be surprising given the result of Exercise 2.9.7. Furthermore, this result directly follows from the general theorem 2.9.3.

2.10 Geodesics and normal coordinates

Let (M, g) be an n-dimensional (pseudo-)Riemannian manifold with Levi-Civita connection ∇.

Definition 2.10.1. *A curve $c : I \subset \mathbb{R} \to M$ is called a **geodesic** if $\dot{c}(t)$ is parallel along c,*

$$\nabla_{\dot{c}(t)} \dot{c}(t) = 0, \tag{2.252}$$

for all $t \in I$.

Some remarks.

- From Eq. (2.162), we recall that in local coordinates of a chart (U, ϕ), the component functions $x^i = (\phi^i \circ c)$, where ϕ^i denotes the restriction onto the i-th coordinate, $\dot{c}(t) = \frac{\mathrm{d}x^i}{\mathrm{d}t} \partial_i = \dot{x}^i \partial_i$ must thus satisfy

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0. \tag{2.253}$$

Existence and uniqueness of such a geodesic, given $c(0)$ and $\dot{c}(0)$ follows from the existence and uniqueness of solutions to systems of homogeneous linear differential equations (such as Eqs. (2.253)).

- One can show that curves of the form (2.252) (locally) minimize the length functional on (pseudo-)Riemannian manifolds (see Exercise 2.10.6). Hence, such curves represent curves of minimal distance between given points on the manifold.

- From the linearity of Eqs. (2.253) it immediately follows that if $c(t)$ is a geodesic, then $c(\lambda t)$ is a geodesic with initial 'velocity' $\lambda\dot{c}$, where $\lambda \in \mathbb{R}$.

Definition 2.10.2. *Let $p \in M$ and $V \subset T_pM$ be an open neighborhood of $0 \in T_pM$. For $v \in V$ denote c_v the geodesic with $c_v(0) = p$ and $\dot{c}_v(0) = v$. The **exponential map** at p is defined by*

$$\exp_p : V \subset T_pM \quad \to \quad M \tag{2.254}$$

$$v \quad \mapsto \quad c_v(1). \tag{2.255}$$

Theorem 2.10.3. *The exponential map $\exp_p$ at $p \in M$ is a diffeomorphism from a neighborhood V of $0 \in T_pM$ to a neighborhood $U \subset M$ of p.*

Proof. Note that by construction of the exponential map, $\exp_p(tv) = c_{tv}(1) = c_v(t)$, since c is a geodesic (see comment above). Since $c_v(t)$ depends smoothly on its initial conditions, thanks to smooth dependence of Eq. (2.253) on initial conditions, $\exp_p$ is differentiable. Therefore, it induces a differential (tangent) map

$$\mathrm{d}\exp_p : TU \to TV \tag{2.256}$$

with

$$\mathrm{d}_0 \exp_p w = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp_p(tw) = \dot{c}_w(0) = w, \quad w \in V, \tag{2.257}$$

at $0 \in T_pM$. Therefore, $\mathrm{d}_0 \exp_p = \mathrm{Id}$ and $\mathrm{d}_0 \exp_p$ is linear and invertable. The inverse function theorem for smooth functions on manifolds (which we shall not discuss here further) then guarantees the existence of an open neighborhood V of $0 \in T_pM$ on which $\exp_p$ is invertable, with the inverse $\exp_p^{-1}$ also being smooth. Thus we conclude that $\exp_p$ is a diffeomorphism on that open neighborhood. □

Normal Coordinates. Note that due to the preceding theorem, at every point $p \in M$ on the manifold the exponential map provides a chart (U, ϕ) of the manifold around $p \in U$ in the following way:

$$\phi = \exp_p^{-1} : U \subset M \quad \to \quad V \subset T_pM \simeq \mathbb{R}^n \tag{2.258}$$

$$q \quad \mapsto \quad \exp_p^{-1}(q). \tag{2.259}$$

Let $e_1, \ldots, e_n$ denote the canonical basis of T_pM, i.e., the basis with

$$g_p(e_i, e_j) = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1) \tag{2.260}$$

(see Theorem 2.5.2). Let $x^1, \ldots, x^n$ denote the corresponding coordinates on V. Then $\exp(x^i e_i)$ maps V into U, i.e., any $q \in U$ can be identified with corresponding coordinates $x^1, \ldots, x^n$ on V through $\exp_p^{-1}(q) = x^i e_i$ for some $x^1, \ldots, x^n$. For any $v = v^i e_i \in V$, the corresponding geodesic satisfies $c_v(t)$, $c_v(t) = \exp_p(vt) = \exp_p(x^i e_i)$; that is, the coordinates of the geodesic are $x^i = v^i t$ and thus

$$\ddot{x}^i(t) = 0. \tag{2.261}$$

The geodesic equations (2.253) reduce to

$$\Gamma^i_{jk} v^j v^k = 0, \tag{2.262}$$

where the Christoffel symbols are evaluated at p. Since the Christoffel symbols are symmetric in j, k (see Theorem 2.7.3) it follows that they must all vanish at p,

$$\Gamma^i_{jk} = 0 \qquad \text{at } \ p, \ \ \text{for all } i, j, k. \tag{2.263}$$

Exercise 2.10.4 shows that this implies

$$\partial_i g_{jk} = 0 \qquad \text{at } \ p, \ \ \text{for all } i, j, k. \tag{2.264}$$

Furthermore, observe that in these coordinates (cf. Eq. (2.257)),

$$\partial_i|_p = \mathrm{d}_0(\exp_p) e_i = e_i, \tag{2.265}$$

since the differential of the exponential map at $0 \in T_pM$ is the identity map. Thus we conclude that at p:

$$g_{ij} = g(\partial_i|_p, \partial_j|_p) = g_p(e_i, e_j) = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1). \tag{2.266}$$

In particular, if M is a four-dimensional Lorentzian manifold,

$$g_{ij} = \eta_{ij} \qquad \text{at } \ p. \tag{2.267}$$

We call these coordinates **normal coordinates** of M at p.

We have thus shown that for any spacetime a chart can be constructed in which part of the same Minkowski space ($V \subset T_pM$) for $p \in M$ can be mapped onto a region $U \subset M$ on spacetime, such that at p the metric is given by the Minkowski metric; that is, any spacetime admits local coordinates such that it looks locally like Minkowski space. This is a mathematical formulation underlying the equivalence principle, which we shall discuss further in Sec. 3.

A natural question to ask is whether such normal coordinates and the notion of a spacetime that looks locally like Minkowski space can be extended somewhat more globally. At the end of the next section (Sec. 2.11.4), we will show that for a finite neighborhood of a point p this can only be achieved if and only if the manifold is locally flat in such a finite neighborhood, i.e., if and only if the curvature tensor vanishes identically. However, in Sec. 3.7 we will get back to this question and show that, in fact, such normal coordinates at a point p can be extended more globally; they can be transported along a geodesic and thus provide a reference frame that looks like Minkowski space all along this curve. This extension of normal coordinates is called **Fermi normal coordinates**.

Exercise 2.10.4. *Show that Eq. (2.263) implies that the partial derivatives of the components of the metric tensor vanish at p (Eq, (2.264)).*

Exercise 2.10.5. *Show that in flat space, $M = \mathbb{R}^n$, geodesics are straight lines.*

Exercise 2.10.6. ***(Geodesics from variational principle)***

(a) Show that the geodesic equations (2.253) follow from the Euler-Lagrange equation,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0, \tag{2.268}$$

of the Lagrangian $\mathcal{L} = g(\dot{c},\dot{c}) = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$. As usual, the dot denotes differentiation with respect to the parameter λ parametrizing the geodesic curve c. Notice that $\mathcal{L}$ is the squared line element, $\mathcal{L} = \left(\frac{\mathrm{d}s}{\mathrm{d}\lambda}\right)^2$. Geodesics thus extremize the length between given points on M.

(b) Show that $\mathcal{L}$ is constant along a geodesic, i.e., that

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\lambda} = 0. \tag{2.269}$$

This shows that $\mathcal{L} = const.$ is a first integral of the geodesic equation.

Exercise 2.10.7. *Consider $M = \mathbb{R}^2$ with polar coordinates $(x^1, x^2) = (r, \phi)$. The metric is given by*

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\phi^2. \tag{2.270}$$

Compute the Christoffel symbols and show that the geodesic equations (2.253) are given by

$$\ddot{r} - r\dot{\phi}^2 = 0 \tag{2.271}$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0 \tag{2.272}$$

Rederive these equations more easily using Eq. (5.69) from Exercise 2.10.6. Together with condition (2.269) of that exercise, show that the geodesic equations can be summarized as

$$\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + \frac{1}{r^2} = \frac{1}{a^2}, \tag{2.273}$$

where $a = const.$ Verify that the solutions to this equation are, in fact, straight lines, as already shown in Cartesian coordinates in Exercise 2.10.5 (hint: solve introducing $u = 1/r$).

Exercise 2.10.8. ***(Geodesics in Schwarzschild spacetime)*** *Consider the Schwarzschild spacetime $M = \mathbb{R} \times (2m, \infty) \times S^2$ with coordinates (t, r, Θ, ϕ) and metric as defined in Exercise 2.7.6. Consider a geodesic c in M and evaluate the constraint $g(\dot{c},\dot{c}) = -1$ explicitly (which results from parametrizing the geodesic by proper time, as we will see in Sec. 3.2). Use the results from Exercise 2.7.6 to show that the geodesic equations in this spacetime are given by*

$$\ddot{t} = -\frac{2m}{r^2 h(r)}\dot{t}\dot{r}, \tag{2.274}$$

$$\ddot{r} = -\frac{h(r)m}{r^2}\dot{t}^2 + \frac{m}{r^2 h(r)}\dot{r}^2 + rh(r)\dot{\theta}^2 + rh(r)\sin^2\theta\,\dot{\phi}^2, \tag{2.275}$$

$$\ddot{\theta} = \sin\theta\cos\theta\,\dot{\phi}^2 - \frac{2}{r}\dot{r}\dot{\theta}, \tag{2.276}$$

$$\ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi} - 2\cot\theta\,\dot{\theta}\dot{\phi}. \tag{2.277}$$

2.11 Curvature

In this section, we will introduce the concept of curvature as embodied by the Riemannian curvature tensor, and derive conditions for local 'flatness' of a manifold. The manifold (M, g) under consideration here shall be, as usual, an n-dimensional (pseudo-)Riemannian manifold with Levi-Civita connection ∇.

We shall first introduce curvature from an abstract point of view, and then develop a more intuitive interpretation using parallel transport. This section concludes with the local flatness theorem, which provides an unambiguous way of distinguishing between a flat and curved manifold.

2.11.1 The Riemann curvature tensor

Definition 2.11.1. *Let $X, Y, Z \in \mathcal{X}(M)$. The tri-linear map*

$$R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M) \tag{2.278}$$

$$(X, Y, Z) \mapsto R(Y, Z)X, \tag{2.279}$$

where

$$R(Y, Z)X = \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X, \tag{2.280}$$

is called ***curvature*** *or* ***curvature operator****. The corresponding multi-linear map*

$$R : \mathcal{X}^*(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathbb{R} \tag{2.281}$$

$$(\omega, X, Y, Z) \mapsto \omega(R(Y, Z)X) \tag{2.282}$$

is called ***Riemann curvature tensor****. The* ***covariant curvature tensor*** *is defined by the multi-linear map*

$$R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathbb{R} \tag{2.283}$$

$$(V, X, Y, Z) \mapsto g(V, R(Y, Z)X). \tag{2.284}$$

Some remarks.

- The Riemann curvature tensor actually is a tensor field $R \in \mathcal{T}^1_3(M)$. This follows from the fact that R is multi-linear and $\mathcal{F}(M)$-homogeneous, i.e.,

$$R(fX, gY)hZ = fghR(X, Y)Z \quad \text{for all } f, g, h \in \mathcal{F}(M). \tag{2.285}$$

  A well known theorem states that multilinear maps of vector and covector fields on a manifold are tensor fields if and only if they are $\mathcal{F}(M)$-homogeneous.

- The covariant curvature tensor is a $(0, 4)$-tensor field, $R \in \mathcal{T}^0_4(M)$, which can be shown in the same way as for the $(1, 3)$-Riemann curvature tensor field. Note that the covariant curvature tensor can be obtained from the Riemann curvature tensor by lowering the first index (see Sec. 2.5 on raising and lowering indices).

Exercise 2.11.2. *Show that the Riemann curvature tensor actually is a* $(1,3)$*-tensor field on* M*, i.e., show that Eq* (2.285) *holds. Analogously, show that the covariant curvature tensor is a* $(0,4)$*-tensor field.*

Theorem 2.11.3. *The components of the Riemann curvature tensor and the covariant curvature tensor in a chart* (U,ϕ) *of* M *are given by*

$$R^r_{ijk} = \partial_j \Gamma^r_{ki} - \partial_k \Gamma^r_{ji} + \Gamma^r_{js}\Gamma^s_{ki} - \Gamma^r_{ks}\Gamma^s_{ji}, \tag{2.286}$$

$$R_{rijk} = g_{rs} R^s_{ijk}. \tag{2.287}$$

respectively.

Proof. Let $\{\partial_i\}$ and $\{\mathrm{d}x^i\}$ denote the basis vector and covector fields with respect to ϕ. We start by computing

$$\begin{aligned}
R(\partial_j,\partial_k)\partial_i &= \nabla_{\partial_j}\nabla_{\partial_k}\partial_i - \nabla_{\partial_k}\nabla_{\partial_j}\partial_i && (2.288)\\
&= \nabla_{\partial_j}(\Gamma^m_{ki}\partial_m) - \nabla_{\partial_k}(\Gamma^m_{ji}\partial_m) && (2.289)\\
&= (\partial_j\Gamma^m_{ki})\partial_m + \Gamma^m_{ki}\Gamma^n_{jm}\partial_n - (\partial_k\Gamma^m_{ji})\partial_m - \Gamma^m_{ji}\Gamma^n_{km}\partial_n && (2.290)\\
&= \left[\partial_j\Gamma^n_{ki} - \partial_k\Gamma^n_{ji} + \Gamma^m_{ki}\Gamma^n_{jm} - \Gamma^m_{ji}\Gamma^n_{km}\right]\partial_n. && (2.291)
\end{aligned}$$

Then

$$R^r_{ijk} = \mathrm{d}x^r(R(\partial_j,\partial_k)\partial_i) = \partial_j\Gamma^r_{ki} - \partial_k\Gamma^r_{ji} + \Gamma^m_{ki}\Gamma^r_{jm} - \Gamma^m_{ji}\Gamma^r_{km}. \tag{2.292}$$

The identity for the components of the covariant curvature tensor is trivial, as the covariant curvature tensor is a contraction of the Riemann tensor, and thus follows from the transformation properties of tensor fields (Eq. (2.113)). However, writing it out explicitly, we find that

$$R_{rijk} = R(\partial_r,\partial_i,\partial_j,\partial_k) = g(\partial_r, R(\partial_j,\partial_k)\partial_i) = g(\partial_r, R^s_{ijk}\partial_s) = g(\partial_r,\partial_s)R^s_{ijk} = g_{rs}R^s_{ijk}. \tag{2.293}$$

□

Exercise 2.11.4. *Show that for flat space,* $M = \mathbb{R}^n$*, the Riemann curvature tensor vanishes.*

Since the covariant curvature tensor represents a contraction of the Riemann curvature tensor, we will simply refer to both as the curvature tensor or the Riemann curvature tensor in the following. The curvature tensor has n^4 components, where n is the dimension of M. However, not all of these components are independent due to a number of symmetries, which we shall list below.

Theorem 2.11.5. *Let* $V, X, Y, Z \in \mathcal{X}(M)$ *denote vector fields on* M*. The following identities hold:*

(i) $R(Y,Z)X = -R(Z,Y)X$,

(ii) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ ***(first Bianchi-identity)***,

(iii) $(\nabla_X R)(Y,Z)V + (\nabla_Y R)(Z,X)V + (\nabla_Z R)(X,Y)V = 0$ ***(second Bianchi-identity)***,

(iv) $R(V,X,Y,Z) = -R(V,X,Z,Y)$,

(v) $R(V,X,Y,Z)+R(V,Y,Z,X)+R(V,Z,X,Y) = 0$,

(vi) $R(V,X,Y,Z)=-R(X,V,Y,Z)$,

(vii) $R(V,X,Y,Z)=R(Y,Z,V,X)$,

In components:

(i) $R^r_{ijk}=-R^r_{ikj}$,

(ii) $R^r_{ijk}+R^r_{jki}+R^r_{kij}=0$ ***(first Bianchi-identity)***„

(iii) $R^i_{jkl;m}+R^i_{jlm;k}+R^i_{jmk;l}=0$ ***(second Bianchi-identity)***,

(iv) $R_{rijk}=-R_{rikj}$,

(v) $R_{rijk}+R_{rjki}+R_{rkij}=0$,

(vi) $R_{rijk}=-R_{irjk}$,

(vii) $R_{rijk}=R_{jkri}$.

Proof. Since $[Y,Z]=-[Z,Y]$, (i) can be read off directly from Def. 2.11.1. (ii) follows from the Jacobi identity (Eq. (2.53)): Since ∇ is torsion-free (cf. Theorem 2.6.2), we find:

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y \tag{2.294}$$
$$= \nabla_X(\nabla_Y Z-\nabla_Z Y)+\nabla_Y(\nabla_Z X-\nabla_X Z)+\nabla_Z(\nabla_X Y-\nabla_Y X) \tag{2.295}$$
$$-\nabla_{[X,Y]}Z-\nabla_{[Y,Z]}X-\nabla_{[Z,X]}Y \tag{2.296}$$
$$= \nabla_X[Y,Z]-\nabla_{[Y,Z]}X+\nabla_Y[Z,X]-\nabla_{[Z,X]}Y+\nabla_Z[X,Y]-\nabla_{[X,Y]}Z \tag{2.297}$$
$$= [X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0. \tag{2.298}$$

In order to show (iii), we first note that according to Theorem 2.9.4,

$$(\nabla_X R)(Y,Z)V=\nabla_X(R(Y,Z)V)-R(\nabla_X Y,Z)V-R(Y,\nabla_X Z)V-R(Y,Z)\nabla_X V. \tag{2.299}$$

Taking the cyclic sum of this equation as required by (iii), the two middle terms give rise to the following terms:

$$-[R(\nabla_X Y,Z)+R(Y,\nabla_X Z)+R(\nabla_Y Z,X)+R(Z,\nabla_Y X)+R(\nabla_Z X,Y)+R(X,\nabla_Z Y)] \tag{2.300}$$
$$= -[R(\nabla_X Y,Z)+R(Z,\nabla_Y X)+\text{c.p.}] \tag{2.301}$$
$$= -R([X,Y],Z)+\text{c.p.} \tag{2.302}$$

In the first step, we have simply rearranged terms. In the second step, we have first employed (i), and then made use of the linearity of R together with identity (v) of Theorem 2.6.2. Using the definition of R, the cyclic sum of Eq. (2.299) thus becomes:

$$\nabla_X(R(Y,Z))-R(Y,Z)\nabla_X-R([X,Y],Z)+\text{c.p.} \tag{2.303}$$
$$= \nabla_X(\nabla_Y\nabla_Z-\nabla_Z\nabla_Y-\nabla_{[Y,Z]}) \tag{2.304}$$
$$-(\nabla_Y\nabla_Z-\nabla_Z\nabla_Y-\nabla_{[Y,Z]})\nabla_X \tag{2.305}$$
$$-(\nabla_{[X,Y]}\nabla_Z-\nabla_Z\nabla_{[X,Y]}+\nabla_{[[X,Y],Z]})+\text{c.p.} \tag{2.306}$$

The very last term vanishes in the cyclic sum due to the Jacobi identity; all other terms vanish pairwise in the cyclic sum, and (*iii*) follows.

Note that (*iv*) and (*v*) are just reformulations of (*i*) and (*ii*). Regarding (*vi*), we start by showing that $R(X,X,Y,Z) = 0$. Using the definition of the curvature tensor and the Ricci-identity of the connection (Theorem 2.6.2),

$$R(X,X,Y,Z) = g(X,R(Y,Z)X) = g(X,\nabla_Y\nabla_Z X) - g(X,\nabla_Z\nabla_Y X) - g(X,\nabla_{[Y,Z]}X) = 0 \tag{2.307}$$

This follows from repeatedly using the Ricci-identity,

$$\begin{aligned} g(X,\nabla_Y\nabla_Z X) &= Yg(X,\nabla_Z X) - g(\nabla_Y X,\nabla_Z X) && (2.308)\\ &= \frac{1}{2}YZg(X,X) - g(\nabla_Y X,\nabla_Z X), && (2.309)\end{aligned}$$

$$\begin{aligned} g(X,\nabla_Z\nabla_Y X) &= Zg(X,\nabla_Y X) - g(\nabla_Z X,\nabla_Y X) && (2.310)\\ &= \frac{1}{2}ZYg(X,X) - g(\nabla_Z X,\nabla_Y X), && (2.311)\end{aligned}$$

$$g(X,\nabla_{[Y,Z]}X) = \frac{1}{2}[Y,Z]g(X,X), \tag{2.312}$$

and substituting these expressions back into Eq. (2.307), in which all terms cancel. Now, due to linearity,

$$R(V+X,V+X,Y,Z) = R(V,V,Y,Z)+R(X,X,Y,Z)+R(V,X,Y,Z)+R(X,V,Y,Z). \tag{2.313}$$

According to Eq. (2.307) the first three terms vanish and (*vi*) immediately follows. Identity (*vii*) can be obtained from the previous ones. Identities (*iv*) and (*v*) imply:

$$R(V,X,Y,Z) = -R(V,X,Z,Y) = R(V,Z,Y,X) + R(V,Y,X,Z). \tag{2.314}$$

Identities (*v*) and (*iv*) imply:

$$R(V,X,Y,Z) = -R(X,V,Y,Z) = R(X,Y,Z,V) + R(X,Z,V,Y). \tag{2.315}$$

Therefore,

$$2R(V,X,Y,Z) = R(V,Z,Y,X) + R(V,Y,X,Z) + R(X,Y,Z,V) + R(X,Z,V,Y). \tag{2.316}$$

Changing the pairs $V,X \leftrightarrow Y,Z$, and then using (*iv*) and (*vi*) simultaneously, we find:

$$\begin{aligned} 2R(Y,Z,V,X) &= R(Y,X,V,Z) + R(Y,V,Z,X) + R(Z,V,X,Y) + R(Z,X,Y,V) && (2.317)\\ &= R(V,Z,Y,X) + R(V,Y,X,Z) + R(X,Y,Z,V) + R(X,Z,V,Y) && (2.318)\\ &= 2R(V,X,Y,Z). && (2.319)\end{aligned}$$

□

Remark on index notation. Because of symmetries such as those proven in the preceding theorem, the exact ordering of indices of a tensor does matter. We will henceforth use white spaces where necessary to unambiguously indicate which indices are contra- versus covariant, and precisely in which order.

Independent components of the Riemann tensor. Due to the symmetries discussed in the previous theorem, not all of the n^4 components of the Riemann curvature tensor are independent. Due to the anti-symmetry in the first two and the second two indices (cf. (iv) and (vi) in Theorem 2.11.5), there are $N = n(n-1)/2$ independent components per block index. Due to condition (vii), the components of the Riemann tensor can be identified as a symmetric $N \times N$-matrix with two block indices and

$$\frac{1}{2}N(N+1) = \frac{n(n-1)(n^2-n+2)}{8} \tag{2.320}$$

independent components. Another condition on the number of independent components is imposed by the first Bianchi identity ((v) in Theorem 2.11.5). Using the symmetries (iv), (v), and (vii) we can replace each term in (v) by a term of the form

$$R_{rijk} = \frac{1}{8}(R_{rijk} - R_{irjk} - R_{rikj} + R_{irkj} + R_{jkri} - R_{jkir} - R_{kjri} + R_{kjir}). \tag{2.321}$$

This leads to a representation of (v) in terms of a total anti-symmetric sum, which shows that (v) only constrains the number of independent components if all four indices are pairwise different. Since there are

$$\begin{pmatrix} n \\ 4 \end{pmatrix} = \begin{cases} \frac{n!}{(n-4)!4!}, & n \geq 4 \\ 0, & n < 4 \end{cases} \tag{2.322}$$

possibilities to choose four different indices out of n, the total number of independent components of the Riemann curvature tensor is given by

$$\frac{n(n-1)(n^2-n+2)}{8} - \begin{pmatrix} n \\ 4 \end{pmatrix} = \frac{n^2(n^2-1)}{12}. \tag{2.323}$$

In particular, for two-dimensional manifolds ($n = 2$) there is only one independent component:

$$R_{1212} = -R_{2112} = -R_{1221} = R_{2121}, \tag{2.324}$$

which entirely determines the curvature of such manifolds.

Exercise 2.11.6. ***(The Weyl tensor)***
Let (M, g) be an n-dimensional (pseudo-) Riemannian manifold. The Weyl tensor $C^\alpha_{\beta\mu\nu}$ is defined as the traceless part of the Riemann tensor; it has the same symmetries as the Riemann tensor.

(a) Determine the Weyl tensor by first writing down the most general combination of $R_{\mu\nu}$ and $g_{\mu\nu}$ that can be added to $R_{\alpha\beta\mu\nu}$ without violating the symmetries of $R_{\alpha\beta\mu\nu}$, and then proceed in the same way with a term quadratic in the metric $g_{\mu\nu}$. Now impose the traceless condition $C^\alpha_{\mu\alpha\nu} = 0$ to fix the coefficients of the most general linear combination considered above, and thus find the Weyl tensor. What is the significance of the Weyl tensor in vacuum (i.e., $R_{\mu\nu} = 0$)?

(b) Show that the Weyl tensor is invariant under ***conformal transformations***

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}, \tag{2.325}$$

i.e., $C^\alpha_{\mu\alpha\nu} = \bar{C}^\alpha_{\mu\alpha\nu}$.

2.11.2 Geometric interpretation—curvature and parallel transport

The following theorem provides a geometric interpretation of the curvature operator in terms of parallel transport. In essence, the curvature operator measures the change of a tangent vector as it is parallel transported around an infinitesimal closed loop on the manifold. We start by showing a useful property of the Riemann curvature operator.

Lemma 2.11.7. *Let $p \in M$, $c : U \subset \mathbb{R}^2 \to M$ a two-parameter family of curves in M with $c(0,0) = p$, and $X \in \mathcal{X}(M)$ a smooth vector field. Then:*

$$R\left(\frac{\partial c}{\partial s}, \frac{\partial c}{\partial t}\right) X = \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} X - \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} X, \tag{2.326}$$

where $\partial c/\partial s$ and $\partial c/\partial t$ denote the tangent vectors of c in s and t direction, respectively.

Proof. Let (U^ϕ, ϕ) be a chart of M with coordinate basis vector fields $\{\partial_i\}$, and without loss of generality, let us assume that $U \subset U^\phi$. Then we can write

$$\frac{\partial c}{\partial s} = A^i \partial_i \circ c, \quad A^i = \frac{\partial}{\partial s}(\phi^i \circ c) \tag{2.327}$$

$$\frac{\partial c}{\partial t} = B^i \partial_i \circ c, \quad B^i = \frac{\partial}{\partial t}(\phi^i \circ c), \tag{2.328}$$

where ϕ^i denotes, as usual, the projection onto the i-th coordinate. Furthermore, using the properties of the covariant derivative,

$$\nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} X = \nabla_{\frac{\partial}{\partial s}} (B^i \nabla_{\partial_i} X) \tag{2.329}$$

$$= \frac{\partial B^i}{\partial s}(B^i \nabla_{\partial_i} X) + B^i A^j \nabla_{\partial_j} \nabla_{\partial_i} X. \tag{2.330}$$

Thus

$$\left(\nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} - \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}}\right) X = \left(\frac{\partial B^i}{\partial s} - \frac{\partial A^i}{\partial t}\right) (\nabla_{\partial_i} X) \tag{2.331}$$

$$+ B^i A^j \left(\nabla_{\partial_j} \nabla_{\partial_i} X - \nabla_{\partial_i} \nabla_{\partial_j} X\right) \tag{2.332}$$

$$= B^j A^j R\left(\partial_j, \partial_i\right) X = R\left(\frac{\partial c}{\partial s}, \frac{\partial c}{\partial t}\right), \tag{2.333}$$

where we have used the fact that partial derivatives commute, that the Lie bracket of coordinate vector fields vanishes (Theorem 2.3.7), and the fact that R is $\mathcal{F}$-homogeneous (Exercise 2.11.2). □

Theorem 2.11.8. *Let $p \in M$, $c : U \subset \mathbb{R}^2 \to M$ a two-parameter family of curves in M with $c(0,0) = p$. For any (fixed) s or t let us denote the curves parametrized by t and s by $c_s(t) = c(s,t)$ or $c^t(s) = c(s,t)$, respectively. Furthermore, let $v_0 \in T_pM$ be a tangent vector and*

$$v(s,t) = \tau^{c_0}_{0,t} \circ \tau^{c^t}_{0,s} \circ \tau^{c_s}_{t,0} \circ \tau^{c^0}_{s,0}(v_0) \tag{2.334}$$

the corresponding vector in T_pM that has been parallel transported once along a closed loop along the directions of $c(s,t)$. Then:

$$\lim_{s,t \to 0} \frac{v(s,t) - v_0}{st} = R(\dot{c}(0,0), c'(0,0))v_0, \tag{2.335}$$

where the dot and prime refer to differentiation wrt. t and s, respectively (i.e., to the tangent vectors in t and s direction).

Proof. We start by defining the auxiliary field

$$V(s,t) = \tau^{c_s}_{t,0} \circ \tau^{c^0}_{s,0}(v_0). \tag{2.336}$$

Thus

$$v(s,t) = \tau^{c_0}_{0,t} \circ \tau^{c^t}_{0,s}(V(s,t)). \tag{2.337}$$

With this notation, we note that with the help of Lemma 2.11.7:

$$\nabla_{\frac{\partial}{\partial t}} V = 0 \quad \Leftrightarrow \quad \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} V = R\left(\frac{\partial c}{\partial t}, \frac{\partial c}{\partial s}\right) V. \tag{2.338}$$

Therefore, using Theorem 2.8.4, we find

$$\lim_{s\to 0} \frac{v(s,t) - v_0}{s} = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \tau^{c_0}_{0,t} \circ \tau^{c^t}_{0,s}(V(s,t)) = \tau^{c_0}_{0,t}\left(\nabla_{\frac{\partial}{\partial s}} V\big|_{(0,t)}\right). \tag{2.339}$$

Consequently, making use of the identity Eq. (2.338),

$$\lim_{t,s\to 0} \frac{v(s,t) - v_0}{ts} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tau^{c_0}_{0,t}\left(\nabla_{\frac{\partial}{\partial s}} V\big|_{(0,t)}\right) = \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} V\big|_{(0,0)} \tag{2.340}$$

$$= R\left(\frac{\partial c}{\partial t}, \frac{\partial c}{\partial s}\right) v_0. \tag{2.341}$$

□

Essentially as a corollary of the previous theorem, one obtains without much additional work the following result, which we shall state here without proof:

Theorem 2.11.9. *Let $U \subset M$ be open and simply connected. Then the following statements are equivalent:*

(i) $R \equiv 0$ on U

(ii) For any closed curve $c : [0,1] \to U$ on U with $c(0) = c(1) = p$: $\tau^c_{c(1),c(0)} = \mathrm{Id}_{T_pM}$.

(iii) Parallel transport on U is path-independent.

2.11.3 The Ricci tensor and scalar curvature

The following definition introduces two contractions of the Riemann curvature tensor that are of fundamental importance to the Einstein field equations.

Definition 2.11.10. *Let $\mathrm{Rie} \in \mathcal{T}^1_3(M)$ denote the Riemann curvature tensor on the (pseudo-)Riemannian manifold M. Its contraction*

$$\mathrm{Ric} \equiv C^1_2 \mathrm{Rie} \in \mathcal{T}^0_2(M) \tag{2.342}$$

*is called the **Ricci tensor**. It has components (cf. Def. 2.4.7, Eq. (2.89), and Exercise 2.4.8)*

$$R_{ij} = R^k{}_{ikj}, \tag{2.343}$$

where $R^i{}_{jkl}$ are the components of the Riemann tensor. The contraction of the Ricci tensor,

$$R \equiv C^0_1 \mathrm{Ric} \in \mathcal{F}(M), \tag{2.344}$$

is called the ***scalar curvature****. It is a scalar function, given by (cf. Def. 2.4.7, Eq. (2.89), and Exercise 2.4.8)*

$$R = R^i{}_i. \tag{2.345}$$

Some remarks.

- In local coordinates, the Ricci tensor is written as

$$\mathrm{Ric} = R_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j. \tag{2.346}$$

  One may choose this relation as a starting point to define the Ricci tensor, with the addition that the components are given by $R_{ij} = R^k{}_{ikj}$. This is, in fact, a standard approach. One then needs to show that the Ricci tensor $\mathrm{Ric(X,Y)}$ is linear in $X, Y \in \mathcal{X}(M)$ and $\mathcal{F}$-homogeneous, which is guaranteed in a given chart by construction (it is written as a linear combination of a local 0-2 tensor basis). What is less clear is that this local definition gives rise to a global tensor field, i.e., that this definition is independent of the chart used. This, however, follows from the fact that the components $R^k{}_{ikj}$ are the components of the Riemann tensor, so R_{ij} as a sum of these components transform according to Theorem 2.4.5 and are thus independent of the chart used. Hence, Ric as defined by Eq. (2.346) is indeed a $\mathcal{T}^0_2(M)$ tensor field.

- The scalar curvature is a real-valued function $R \in \mathcal{F}(M)$. This being the contraction of a tensor makes its value at given $p \in M$ independent of the choice of coordinates—it is therefore a **curvature invariant**.

- For two-dimensional manifolds, one can show that the scalar curvature R equals twice the Gaussian curvature and completely determines the curvature properties of the manifold (see Exercises 2.11.13 and 2.11.14 below).

We will now show an identity that will turn out to be central to energy-momentum conservation of Einstein's field equations.

Theorem 2.11.11. *Let R_{ij} and R denote the Ricci tensor and scalar curvature on M, respectively. The Ricci tensor is symmetric and satisfies the contracted Bianchi identity:*

(i) $R_{ij} = R_{ji}$
(ii) $\left(R_i{}^k - \frac{1}{2}\delta^k_i R\right)_{;k} = 0$ ***(contracted Bianchi-identity)****,*

Proof. Regarding (i): Using (vii) of Theorem 2.11.5 and the symmetry of the metric tensor g, we find:

$$R_{ji} = R^k{}_{jki} = g^{kl} R_{ljki} = g^{kl} R_{kilj} = g^{lk} R_{kilj} = R^k{}_{ikj} = R_{ij}. \tag{2.347}$$

Regarding (ii), we start by computing

$$\begin{aligned}
R_i{}^k{}_{;k} &= g^{kl} R_{il;k} = g^{kl} g^{jm} R_{jiml;k} && (2.348)\\
&= g^{kl} g^{jm} R_{mlji;k} && (2.349)\\
&= -g^{kl} g^{jm} (R_{mlik;j} + R_{mlkj;i}) && (2.350)\\
&= -g^{jm} R_{mi;j} + g^{jm} R_{mj;i} && (2.351)\\
&= -R^k{}_{i;k} + R_{;i} && (2.352)
\end{aligned}$$

where we have used (vii) of Theorem 2.11.5 in the second line, (iii) of Theorem 2.11.5 in the third line, and (i) and (vi) of Theorem 2.11.5 in the fourth line. Note that we have also made repeated use of the fact that the covariant derivative of tensors commutes with contractions (Theorem 2.9.3). Therefore,

$$R_i{}^k{}_{;k} = \frac{1}{2} R_{;i} = \frac{1}{2} \delta_i^k R_{;k}. \qquad (2.353)$$

□

Exercise 2.11.12. *Consider the Schwarzschild spacetime from Exercise 2.7.6. Using the expressions computed there for the covariant derivatives, compute the components of the curvature operator. Note that due to symmetries, only the components of the following matrices need to be computed:* (R^k_{i01}), (R^k_{i02}), (R^k_{i03}), (R^k_{i12}), (R^k_{i13}), *and* (R^k_{i23}).

Exercise 2.11.13. ***(Two-dimensional manifolds I)***
Show that for two-dimensional manifolds the components of the curvature tensor can be written as

$$R_{rijk} = (g_{rj} g_{ik} - g_{rk} g_{ij}) \frac{R_{1212}}{\det(g_{lm})}. \qquad (2.354)$$

Compute the components of the Ricci tensor and the scalar curvature.

Exercise 2.11.14. ***(Two-dimensional manifolds II)***
Derive a relation between the scalar curvature R and the ***Gaussian curvature*** $K = 1/(\rho_1 \rho_2)$ *of a two-dimensional manifold M, where ρ_1 and ρ_2 are the local principal curvature radii.*
Hint: Without loss of generality one can locally consider the two-dimensional manifold $M \subset \mathbb{R}^3$, defined by the function

$$z(x, y) = \frac{x^2}{2\rho_1} + \frac{y^2}{2\rho_2}. \qquad (2.355)$$

Determine the line element $\mathrm{d}s^2$ in Cartesian coordinates and identify the metric elements. Then compute the curvature scalar R at $x = y = 0$.
(The idea here is that any two-dimensional surface can locally be described by the paraboloid given here [one can obtain the normal paraboloid, the hyperbolic paraboloid, or the plane, depending on the values and signs of ρ_1 and ρ_2, and any surface can be locally classified according to these categories]. It is therefore sufficient to show that the relation between scalar curvature and Gaussian curvature holds at $x = y = 0$ for this model. Since any surface can be locally approximated by such a paraboloid, the relation between R and K holds globally.)

2.11.4 Local flatness

Intuitively, we think of 'flat space' as $\mathbb{R}^n$. Let us now state this intuitive notion of 'flat space' in more concrete terms.

Definition 2.11.15. *A (pseudo-)Riemannian manifold (M, g) is said to be locally flat at $p \in M$ if there exists a chart (U, ϕ) around p, such that*

$$g = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1). \tag{2.356}$$

everywhere on U.

The fact that the Riemann curvature tensor indeed captures all aspects of curvature is reflected by the following theorem.

Theorem 2.11.16. *A (pseudo-)Riemannian manifold (M, g) is locally flat if and only if the Riemann curvature tensor locally vanishes.*

Proof. If M is locally flat around $p \in M$, the metric g is of the form (2.356) on an open neighborhood U around p. Then all Christoffel symbols (and their derivatives) vanish (cf. Eq. (2.135)) on U and thus also the Riemann curvature tensor (cf. Eq. (2.286)).

Now assume that the Riemann curvature tensor vanishes on an open neighborhood U around p. Let us assume without loss of generality that U is simply connected. Then Theorem 2.11.9 states that parallel transport is independent of the path used if and only if the Riemann curvature tensor vanishes. Therefore, we can construct local basis vector fields $\{e_i\}$ on U by parallel transporting a basis $\{e_i|_p\}$ of T_pM, i.e., we also have vanishing covariant derivatives,

$$\nabla_X e_i = 0. \tag{2.357}$$

for $X \in \mathcal{X}(M)$. Let $\{\alpha^i\}$ denote the corresponding covector basis fields on U. The exterior derivative is then given by

$$\mathrm{d}\alpha^i(e_j, e_k) = -\alpha^i([e_j, e_k]) = 0, \tag{2.358}$$

since according to Theorem 2.6.2 and Eq. (2.357) we must also have

$$[e_j, e_k] = \nabla_{e_j} e_k - \nabla_{e_k} e_j = 0. \tag{2.359}$$

Hence, all α^i are closed forms, $\mathrm{d}\alpha^i = 0$. The Lemma of Poincaré states that these closed forms are also exact locally, i.e., there exist functions $x^i : U \to \mathbb{R}$ with $\alpha^i = \mathrm{d}x^i$. If we chose these x^i as coordinates on U, we have $\partial_i = e_i$ and

$$0 = \nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k \quad \Rightarrow \quad \Gamma^k_{ij} = 0. \tag{2.360}$$

on U due to Eq. (2.357). Equation (2.135) then implies that the metric components are constant on U; with a suitable coordinate transformation one can transform these metric components into the normal form Eq. (2.356). □

Chapter 3

The power of the equivalence principle: physics in curved spacetime

3.1 The equivalence principle revisited

The preceding chapter on differential geometry and (pseudo-)Riemannian manifolds allows us to restate the equivalence principle (Sec. 1.2) in more mathematical terms.

The existence of normal coordinates on a (pseudo-)Riemannian manifold (see Sec. 2.10) states that, locally, the Christoffel symbols and complicated metric components can be 'transformed away'. That is, for a four-dimensional Lorentzian manifold, the metric can be brought into the form of the Minkowski metric locally (at a given point p). Theorem 2.11.16 about local flatness of (pseudo-)Riemannian manifolds shows that this is indeed only possible in infinitesimal regions of such a manifold, unless the manifold is flat in a finite region around that point.

The mathematical structure of a four-dimensional Lorentzian manifold thus seems to mathematically embody Einstein's Equivalence Principle (EEP; Sec. 1.2). The latter states that in arbitrary gravitational fields there exist local inertial frames (freely falling nonrotating systems) in which the laws of special relativity apply, i.e., that gravity can be locally 'transformed away'. However, this should only be true in infinitesimal regions of space and time. We therefore arrive at the following postulate, which represents a mathematical formulation of the EEP:

Postulate 3.1.1. ***Spacetime*** *in general relativity is a four-dimensional Lorentzian manifold* (M, g)*—the mathematical concept of a spacetime as defined in Def. 2.5.8. The metric components* $g_{\mu\nu}$ *give rise to curvature as 'measured' by the Riemann curvature tensor, which distinguishes a Lorentzian manifold from the flat Minkowski spacetime of special relativity, i.e., from the absence of gravitational fields; the metric components are thus interpreted as the* ***gravitational potentials****. Normal coordinates of* M *at* $p \in M$ *are identified as* ***local inertial systems*** *or* ***freely falling frames*** *at* p*, in which gravity (curvature) has been 'transformed away' and the laws of special relativity apply.*

This mathematical formulation of the EEP immediately implies rules for how to find laws of physics in general frames on spacetime.

Principle of general covariance. Since all charts of a manifold represent the same (differentiable) structure, laws of physics should not distinguish between coordinate systems, they ought to hold on any chart of an atlas. In other words, laws of physics in curved spacetime of GR must hold in any coordinates and must thus be formulated in a coordinate-independent way—they must be **covariant** with respect to changes of charts, i.e., to any **smooth coordinate transformations**, which is usually referred to as **general covariance**. In more precise terms, a system of equations expressing laws of physics must be covariant with respect to the group of coordinate diffeomorphisms, which means that

- all quantities in the equations preserve the group structure under group transformations, i.e., transformed quantities exist and they respect associativity of successive coordinate transformations
- both transformed and original quantities satisfy the same set of equations.

It appears that if we formulate laws of physics in terms of equations for smooth scalar, vector, or tensor fields on spacetime, the principle of general covariance is satisfied by construction.

Principle of correspondence. In order to respect the equivalence principle and the existence of local inertial frames through normal coordinates, any law of physics must reduce to the (known) special relativistic form at the origin of such frames. For example, the known laws of special relativistic dynamics and electrodynamics must hold locally. Later we will add to this principle the requirement that the theory must also reduce to the Newtonian theory of gravity in the weak-field limit (Sec. 4.1).

Principle of minimal coupling. In order to satisfy the correspondence principle, laws of physics must only contain quantities that are also present in special relativity, apart from the metric and its derivatives. This, however, still leaves the freedom to introduce additional terms that only depend on curvature and thus vanish locally in normal coordinates or in the special relativistic limit (Theorem 2.11.16). For example, energy-momentum conservation could be expressed in the general theory by

$$\partial_\nu T^{\mu\nu} = 0 \quad \to \quad \nabla_\nu T^{\mu\nu} = 0, \tag{3.1}$$

or

$$\partial_\nu T^{\mu\nu} = 0 \quad \to \quad \nabla_\nu T^{\mu\nu} + g^{\delta\epsilon} R^{\mu}{}_{\delta\eta\nu} \nabla_\epsilon T^{\eta\nu} = 0, \tag{3.2}$$

or by expressions including even higher-order terms in the curvature tensor. In order to avoid such ambiguities, one must additionally require that no additional terms explicitly containing the curvature tensor be introduced in generalizing laws of physics from special relativity to the general theory. This is known as the principle of minimal coupling in general relativity.

Conclusion: substitution rules. It appears that in order to satisfy the equivalence principle and the above mentioned principles that it gives rise to, one can obtain the general versions of laws of physics from their special relativistic forms by making the following substitutions:

$$\eta_{\mu\nu} \quad \to \quad g_{\mu\nu}, \tag{3.3}$$

$$\partial_\mu \quad \to \quad \nabla_\nu, \tag{3.4}$$

$$S_{\mathrm{SR}}, X^\mu_{\mathrm{SR}}, T^{i_1\ldots i_r}_{j_1\ldots j_s,\mathrm{SR}} \quad \to \quad S_{\mathrm{GR}}, X^\mu_{\mathrm{GR}}, T^{i_1\ldots i_r}_{j_1\ldots j_s,\mathrm{GR}} \tag{3.5}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. The values of physical scalar (S), vector (X^μ), and tensor ($T^{i_1\dots i_r}_{j_1\dots j_s}$) fields in the generalized theory are obtained (defined) by coordinate transformation of their special relativistic analogues in the local inertial frame.

This highlights the power of the equivalence principle: the general form of laws of physics in the presence of gravitational fields can be obtained from the special relativistic version by simple substitution rules without knowing the full theory yet! That is, we have not discussed the field equations yet, i.e., what actually causes curvature and gives rise to gravitation. Nonetheless we are able to formulate laws of physics in curved spacetime, *assuming* that curvature is somehow given (determined by yet to be specified relations). We shall apply the principle below and explore some examples in the following sections.

3.2 Proper time and motion of a test body

In special relativity, the proper time τ of an observer moving along a world line $c : I \subset \mathbb{R} \to M$ with coordinates x^μ is given by

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda, \tag{3.6}$$

where λ parametrizes the world line, $c(\lambda)$. Applying the substitution rule (3.3), we immediately obtain the general-relativistic version:

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda. \tag{3.7}$$

According to the equivalence principle, in the local inertial frame at a given $p \in M$ the motion of a test body must obey

$$\left.\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}\right|_p = 0. \tag{3.8}$$

Expanding this expression with the help of the chain rule,

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} = \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\partial_\nu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}, \tag{3.9}$$

one can apply the substitution rule (3.4) and use Eq. (2.243) to obtain the general-relativistic version:

$$\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\partial_\nu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \quad\to\quad \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\nabla_\nu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\nu\delta}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\delta}{\mathrm{d}\lambda}. \tag{3.10}$$

Comparing this with Eq. (2.253), we recognize this as the geodesic equations. Thus we arrive at the important conclusion that test bodies in general relativity move along geodesics, i.e., along curves in spacetime that satisfy (cf. Sec. 2.10)

$$\nabla_{\dot c}\dot c = 0. \tag{3.11}$$

Without loss of generality one can assume a world line to be parametrized by proper time, $c = c(\tau)$. According to Eq. (3.7), the tangent vector $\dot c$, i.e., the four-velocity $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, then satisfies

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = g(\dot c, \dot c) = -1. \tag{3.12}$$

3.3 Gravitational redshift

As one application of the proper time discussed in the previous section, let us consider how the presence of gravitational fields gives rise to a redshift of photons. Let us consider two observers at rest in a static (or stationary) gravitational field, i.e., one observer (A) at the surface of a star that emits light and another observer (B) far away. According to Eq. (3.7), the proper time as measured by a clock along the world line of observer A or B is given by

$$\mathrm{d}\tau = \sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}. \tag{3.13}$$

In general, proper time is influenced by the gravitational field (as encoded through $g_{\mu\nu}$) as well as by motion with respect to the coordinates (as given by $\mathrm{d}x^\mu$). Here, we are solely interested in the effect of the gravitational field on proper time—both observers are at rest ($\mathrm{d}x^i = 0$). Consider an electromagnetic wave emitted at A and received at B. The respective clocks measure the proper time

$$\mathrm{d}\tau_A = \sqrt{-g_{00}(x_A^\mu)}\,\mathrm{d}t_A, \quad \mathrm{d}\tau_B = \sqrt{-g_{00}(x_B^\mu)}\,\mathrm{d}t_B, \tag{3.14}$$

where we have renamed $\mathrm{d}x^0 = \mathrm{d}t$ as the coordinate time on the given chart (U, ϕ) with coordinates x^μ. Let the time interval $\mathrm{d}\tau$ denote the time interval between two consecutive wave crests emitted at A or received at B, which implies that the proper time corresponds to the inverse of the frequencies of the electromagnetic wave as measured by A and B, respectively:

$$\mathrm{d}\tau_A = \frac{1}{\nu_A}, \quad \mathrm{d}\tau_B = \frac{1}{\nu_B}. \tag{3.15}$$

Since the gravitational field is time-independent and both observers are at rest, both wave crests require the same coordinate time to travel from A to B and thus keep their initial time separation, i.e., $\mathrm{d}t_A = \mathrm{d}t_B$. Therefore, combining the above expressions we obtain a frequency or wavelength shift of

$$\frac{\nu_B}{\nu_A} = \frac{\lambda_A}{\lambda_B} = \sqrt{\frac{g_{00}(x_A^\mu)}{g_{00}(x_B^\mu)}}. \tag{3.16}$$

In the case of weak fields, $g_{00} \simeq -1 - 2\Phi/c^2$, where Φ is the Newtonian gravitational potential (see Sec. 4.3), and one finds

$$\frac{\Delta\nu}{\nu} = \frac{\nu_B}{\nu_A} - 1 \simeq \frac{\Phi(x_A^i) - \Phi(x_B^i)}{c^2}. \tag{3.17}$$

Such a redshift manifests itself in shifting atomic lines we observe from distant stars. Einstein originally considered the detection of such a gravitational redshift a crucial test of general relativity. However, it is *solely a consequence of the equivalence principle*, the field equations do not play any role here. It can thus be regarded as a test of the equivalence principle, but not as a test of general relativity itself.

The gravitational redshift of the Sun is minute:

$$\frac{\Delta\nu}{\nu} \simeq \frac{\Phi(x_A^i) - \Phi(x_B^i)}{c^2} \approx \frac{\Phi(x_A^i)}{c^2} = \frac{GM_\odot}{c^2 R_\odot} \approx 2 \times 10^{-6}, \tag{3.18}$$

assuming that at Earth we have essentially climbed the gravitational potential of the Sun ($|\Phi(x_B^i)| \ll \Phi(x_A^i)$). In practice, photons of most atomic lines in the solar spectrum are actually blueshifted, as the dominant effect on photon frequencies is a blueshift caused by large-scale convection (turbulent motion) in the outer convective envelope of the Sun, resulting in $\mathrm{d}x^i_A \neq 0$. These convective motions transport hot gas from the interior to the surface with a large radial velocity component.

Einstein was heavily involved in building a solar telescope in Potsdam, Germany, in the 1920s (nowadays called the ***'Einstein Tower'***), specifically designed to measure the solar gravitational redshift. Unfortunately, not much was known in those days about stellar structure and evolution, and the ***convective blueshift*** prevented them from performing this test of the equivalence principle. Nevertheless, this telescope triggered foundational research on Solar and Stellar Physics.

It was not until 1972 that an experiment measuring the Solar redshift was successful and its results got published (Snider 1972)[1]. Snider used a potassium absorption line in the solar spectrum at 7699 Å, which was believed to be formed high enough in the photosphere so that convective up- and downward motion could not produce a dominating net Doppler effect on the line. It was also shown that this line had no significant center-to-limb effect. Snider was able to measure a ≈ 16 mÅ redshift compared to laboratory measurements on Earth, in agreement with Einstein's predictions. A more precise (1%-level) terrestrial measurement had already been performed in 1965 (Pound & Snider 1965). Both measurements made use of the Mössbauer effect to generate extremely narrow resonance lines to accurately measure a line shift. The most precise redshift measurement to date reached an accuracy of 2×10^4 and was carried out with a hydrogen-maser clock aboard a rocket at 10,000 km altitude in 1976 (Vessot et al. 1980); see Will (2006) for more details on redshift tests.

In general, as noted above, both motion of the source and receiver as well as gravitational fields determine the proper time of observers and the frequency of electromagnetic radiation they send or receive. Clocks aboard GPS satellites, for example, go *faster* with respect to an observer at rest on Earth, because the gravitational field of the Earth is weaker; however, they also go *slower* as the velocity of the satellite is larger. Both effects need to be taken into account accurately. We will explore this in an exercise below.

Finally, we note that in addition to motion and gravitational fields, the ***cosmological redshift*** due to the expansion of the Universe is another source of redshift. We will, however, not discuss it here.

Exercise 3.3.1. *Evaluate Eq. (3.17) for a weak homogeneous gravitational field, which we approximately experience locally on the surface of the Earth. Now consider a single photon and calculate the frequency shift of a photon traveling along the gravitational field just by using energy conservation and the mass-energy equivalence. Compare the two results.*

Exercise 3.3.2. ***(Time synchronization for GPS satellites)***
This problem is to show that for proper time synchronization, GPS satellites need to take both gravitational effects and effects due to motion into account. As we will show in Sec. 7.2 (cf. Eq. (7.27)), a good approximation to the metric outside the surface of the Earth is given by

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{3.19}$$

where

$$\Phi(r) = -\frac{GM_{\text{Earth}}}{r} \tag{3.20}$$

[1]A previous measurement in 1962 by Brault seemed to have remained unpublished (Snider 1972).

is Earth's Newtonian gravitational potential, G the gravitational constant and $M_{\rm Earth}$ is the mass of the Earth. Consider a satellite of mass m orbiting the Earth on a circular equatorial orbit ($\theta = \pi/2$) with radius $r_{\rm sat}$ and velocity $v_{\rm sat}$. Furthermore, let $t_{\rm sat}$ denote the time of the satellite's clock, t_∞ the time a clock would show at spatial infinity $r \to \infty$, and $t_{\rm lab}$ the time shown by a clock on the surface of the Earth.

(*a*) *Determine the relativistic time shift $t_{\rm sat}/t_\infty \simeq 1 + \delta$ due to the motion of the satellite in Earth's gravitational potential to lowest non-vanishing order in v/c and Φ/c^2, and express δ in terms of $\Phi(r_{\rm sat})$ (note that the metric (3.19) is given in geometric units, i.e., velocities are in units of the speed of light c, Φ is in units of c^2).*

(*b*) *Determine $t_{\rm lab}/t_\infty$ as in (a), neglecting the effect due to Earth's rotation (which is much smaller than the satellite's velocity).*

(*c*) *Calculate the relative time shift $(t_{\rm lab} - t_{\rm sat})/t_{\rm lab}$ between Earth and the satellite as a function of $r_{\rm sat}/R$, where R is the radius of the Earth. Determine the sign and order of magnitude of this effect for a low Earth orbit and a geostationary orbit of the satellite.*

3.4 Energy and momentum conservation, relativistic hydrodynamics

As a result of translation invariance, the energy momentum tensor of a closed system satisfies the conservation law

$$\partial_\nu T^{\mu\nu} = 0. \tag{3.21}$$

in special relativity. According to our substitution rules (3.3)–(3.5) the corresponding expression in general relativity is

$$\nabla_\nu T^{\mu\nu} = 0, \tag{3.22}$$

where

$$T^{\mu\nu} = T^{\mu\nu}_{\rm matter} + T^{\mu\nu}_{\rm EM} + \ldots \tag{3.23}$$

is the corresponding generalized energy momentum tensor of special relativity that includes all sources of energy other than gravitation (matter itself, electromagnetic fields, etc.). As one example, consider the energy momentum tensor of an ideal fluid (isotropic fluid as seen by a comoving observer) given by

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p\eta^{\mu\nu} \tag{3.24}$$

in special relativity. Employing the substitution rule (3.3), we can generalize this tensor to general relativity in a straightforward manner:

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}. \tag{3.25}$$

Exercise 3.4.1. ***(General-relativistic Euler Equation)*** *Employ the energy momentum tensor of an ideal fluid, Eq. (3.25), and contract the corresponding energy momentum conservation equation Eq. (3.22) with the projection tensor $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ to obtain the general-relativistic Euler equation:*

$$(\rho + p)\nabla_u u = -\nabla p - (\nabla_u p) u, \tag{3.26}$$

where $\nabla p = ({\rm d}p)^\sharp$.

3.5 Electrodynamics in curved spacetime

The general-relativistic Maxwell equations can be obtained from the special-relativistic version,

$$\partial_\nu F^{\mu\nu} = 4\pi j^\mu, \tag{3.27}$$
$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0, \tag{3.28}$$

by employing the substitution rules (3.3)–(3.5):

$$\nabla_\nu F^{\mu\nu} = 4\pi j^\mu, \tag{3.29}$$
$$\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = 0. \tag{3.30}$$

Here, $F_{\mu\nu}$ is the antisymmetric electromagnetic field tensor, and $j^\mu = (\rho_{\mathrm{e}}, j^i)$ the electric current four-vector. It can be shown that the inhomogeneous Maxwell equation implies the covariant current conservation (see exercise below):

$$\nabla_\mu j^\mu = 0. \tag{3.31}$$

Employing the identities Eq. (2.158) and Eq. (3.37) one can rewrite this equation and the inhomogeneous Maxwell equation as

$$\partial_\nu(\sqrt{|g|}F^{\mu\nu}) = 4\pi\sqrt{|g|}j^\mu, \tag{3.32}$$
$$\partial_\mu(\sqrt{|g|}j^\mu) = 0. \tag{3.33}$$

The electromagnetic energy-momentum tensor can be immediately obtained from the special-relativistic analogue by applying the substitution rules (3.3) and (3.5):

$$T^{\mu\nu}_{\mathrm{EM}} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{3.34}$$

Finally, we derive the general-relativistic equation of motion for a particle of charge q and mass m in an external electromagnetic field. The Lorentz force gives rise to the following equation of motion in special relativity:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = \frac{q}{m}F^{\mu\nu}u_\nu. \tag{3.35}$$

With the help of Eqs. (3.9) and (3.10), we immediately obtain the general-relativistic version:

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\nu\delta}u^\nu u^\delta = \frac{q}{m}F^{\mu\nu}u_\nu. \tag{3.36}$$

The electromagnetic force on the particle is captured by $F^{\mu\nu}$, whereas the gravitational force is encoded in the Christoffel symbols.

Exercise 3.5.1. *Proof Eq.* (3.31)*. To this end, first use the property* (2.157) *to show that for any antisymmetric tensor field,*

$$\nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\nu(\sqrt{|g|}F^{\mu\nu}). \tag{3.37}$$

3.6 Spin precession

Employing the equivalence principle, one can also derive the motion of a spin in curved spacetime. Here, 'spin' refers to the spin of an elementary particle or to the angular momentum of a macroscopic rigid body. This spin, denoted by the vector $\mathbf{S}$, is defined with respect to an inertial frame in which the particle or body is at rest.

Let us first consider the evolution of the spin of an accelerated particle in special relativity, which we will then generalize to curved spacetime. Let us denote the inertial frame in which the particle/body is momentarily at rest by a bar. In the absence of torques,

$$\frac{\mathrm{d}\bar{\mathbf{S}}}{\mathrm{d}t} = 0 \tag{3.38}$$

in this rest frame. In terms of four-vectors, we can define $\bar{S}^\mu = (0, \bar{\mathbf{S}})$ and write

$$\frac{\mathrm{d}\bar{S}^\mu}{\mathrm{d}t} = \left(\frac{\mathrm{d}\bar{S}^0}{\mathrm{d}t}, \frac{\mathrm{d}\bar{\mathbf{S}}}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}\bar{S}^0}{\mathrm{d}t}, \mathbf{0}\right), \tag{3.39}$$

which also holds when parametrized by proper time τ. Furthermore, the particle's three-velocity vanishes, and the four-velocity is given by $\bar{u}^\mu = (-1, \mathbf{0})$ (cf. Eq. (3.6)). This means that in the inertial frame in which the particle or body is momentarily at rest, one has $\eta_{\mu\nu}\bar{u}^\mu \bar{S}^\nu = 0$. The left-hand side of this expression being a Lorentz scalar implies that this equation holds in any inertial frame,

$$\eta_{\mu\nu} u^\mu S^\nu = 0. \tag{3.40}$$

In particular, this identity holds in the lab frame we use to describe the accelerated motion of the object. The evolution of the spin in the lab frame is obtained by a Lorentz transformation Λ^μ_ν from the local rest frame into the lab frame, which yields $u^\mu = \Lambda^\mu_0(-1)$ and

$$\frac{\mathrm{d}S^\mu}{\mathrm{d}\tau} = \Lambda^\mu_\nu \frac{\mathrm{d}\bar{S}^\nu}{\mathrm{d}\tau} = \Lambda^\mu_0 \frac{\mathrm{d}\bar{S}^0}{\mathrm{d}\tau} = -u^\mu \frac{\mathrm{d}\bar{S}^0}{\mathrm{d}\tau} \tag{3.41}$$

Differentiating Eq. (3.40) with respect to proper time, one obtains:

$$\frac{\mathrm{d}S^\mu}{\mathrm{d}\tau} u_\mu = -S^\mu a_\mu, \tag{3.42}$$

where $a^\mu = \mathrm{d}u^\mu/\mathrm{d}\tau$ is the acceleration. Combining Eqs. (3.41) and (3.42), we arrive at

$$\frac{\mathrm{d}S^\mu}{\mathrm{d}\tau} = S^\nu a_\nu u^\mu. \tag{3.43}$$

This equation describes the precession of a spin due to the acceleration of the particle or body, known as **Thomas precession**. Thomas precession is an important correction to the spin-orbit interaction of an electron inside the electric potential of an atomic nucleus. Thomas precession reduces the precession of the electron's spin due to the torque exerted by the magnetic field $\mathbf{B} \approx -\mathbf{v}/c \times \mathbf{E}$ by half.

Let us now find the generalization of Thomas precession in curved spacetime. According to the equivalence principle, Eqs. (3.40) and (3.43) must also hold in the local inertial frame of the particle or body at any given point p in curved spacetime. Applying the substitution rules

Eqs. (3.3) and (3.5), the covariant version of Eq. (3.40) in curved spacetime is straightforwardly obtained:

$$g(S, u) = 0. \tag{3.44}$$

Furthermore, according to the substitution rule (3.4) and property (*i*) of the covariant derivative (cf. Def. 2.6.1), we have

$$\frac{\mathrm{d}S^\mu}{\mathrm{d}\tau} = \partial_\nu S^\mu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \quad \rightarrow \quad u^\nu \nabla_{\partial_\nu} S = \nabla_u S, \tag{3.45}$$

and, analogously,

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} \quad \rightarrow \quad \nabla_u u \equiv a. \tag{3.46}$$

Therefore, the generalized version of Eq. (3.43) in curved spacetime is

$$\nabla_u S = g(S, a)u. \tag{3.47}$$

This leads to spin precession known as **Fermi transport**. Note that this general spin precession equation has three limits:

1. **Gravitational field only.** In the absence of any external force (external acceleration $a = 0$), and noting that $u = \dot{c}$, where c is the particle's world line, we find from Eq. (3.47):

$$\nabla_u S = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}S^\mu}{\mathrm{d}\tau} = -\Gamma^\mu_{\nu\lambda} u^\nu S^\lambda. \tag{3.48}$$

   This describes the precession effect of a spin solely due to the presence of a gravitational field.

2. **No gravitational field.** In this case, Fermi transport (Eq. (3.47)) reduces to the special relativistic Thomas precession (Eq. (3.43)).

3. **Gravitational field plus external force.** This is the full Fermi transport case, i.e., Eq. (3.47) with $a \neq 0$,

$$\frac{\mathrm{d}S^\mu}{\mathrm{d}\tau} = -\Gamma^\mu_{\nu\lambda} u^\nu S^\lambda + g_{\nu\lambda} S^\nu a^\lambda u^\mu. \tag{3.49}$$

   Note that without an external force, a particle or body would be freely falling, i.e., following a geodesic. This would imply $a = \nabla_u u = \nabla_{\dot{c}} \dot{c} = 0$ (cf. Eqs. (2.252) and (3.11)), and Case 1 would apply.

3.7 Fermi transport & Fermi normal coordinates

We now turn back to the question of whether one can extend (pseudo-)Riemannian normal coordinates somewhat globally on a curved spacetime (see Sec. 2.10). It turns out that one can transport normal coordinates at a point p along a geodesic and conserve their properties.

The starting point for this concept referred to as Fermi normal coordinates is the realization that Eqs. (3.44) and (3.47) derived in the previous section give rise to a linear isomorphism between tangent spaces of a Lorentzian manifold, similar to parallel transport:

Definition 3.7.1. *Let $c : I \subset \mathbb{R} \to M$ be a curve on a Lorentzian manifold, parametrized by proper time τ. Then we can write its tangent vector as $\dot{c} = u$, with $g(u,u) = -1$. The* ***Fermi derivative*** *$\mathcal{F}_u$ of a vector field $X \in \mathcal{X}(M)$ along the curve is defined by*

$$\mathcal{F}_u X = \nabla_u X - g(X,a)u + g(X,u)a, \tag{3.50}$$

where $a = \nabla_u u$ is the acceleration. A vector field $X \in \mathcal{X}(M)$ is said to be ***Fermi-transported*** *along c if $\mathcal{F}_u X = 0$.*

Some remarks.

- We note that for the spin vector field S discussed in the previous section, we have

$$\mathcal{F}_u S = 0 \tag{3.51}$$

according to Eqs. (3.44) and (3.47), hence the name Fermi 'transport' for the spin precession equation (3.47).

- It is obvious from the definition that the Fermi derivative is a straightforward generalization of the covariant derivative. This is also reflected by the following properties that immediately follow from the definition:

  (*i*) $\mathcal{F}_u = \nabla_u$ if c is a geodesic ($a = \nabla_u u = \nabla_{\dot{c}} \dot{c} = 0$; cf. Eqs. (2.252) and (3.11))

  (*ii*) $\mathcal{F}_u u = 0$

  (*iii*) If $\mathcal{F}_u X = \mathcal{F}_u Y$ for two vector fields $X, Y \in \mathcal{X}(M)$ along c, then $\mathrm{d}g(X,Y)/\mathrm{d}\tau = 0$, i.e., $g(X,Y)$ is constant along c.

  (iv) If $g(X,u) = 0$ along c for a vector field $X \in \mathcal{X}(M)$, then $\mathcal{F}_u X = (\nabla_u X)_\perp$, where $\perp$ denotes the projection perpendicular to $u = \dot{c}$.

- Since Eq. (3.50) is linear in X, the Fermi derivative defines a linear isomorphism between tangent spaces, in analogy to parallel transport:

$$\begin{aligned} \tau^{\mathcal{F}}_{t,s} : T_{c(s)}M &\to T_{c(t)}M &(3.52)\\ v &\mapsto \tau^{\mathcal{F}}_{t,s} v. &(3.53)\end{aligned}$$

One can show that in analogy to Theorem 2.8.4,

$$\begin{aligned} \mathcal{F}_{\dot{c}} X_{c(t)} &= \lim_{h\to 0} \frac{1}{h}\left(\tau^{\mathcal{F}}_{t,t+h} X_{c(t+h)} - X_{c(t)}\right) &(3.54)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \tau^{\mathcal{F}}_{t,s} X_{c(s)} &(3.55)\\ &\equiv \dot{v}^{\mathcal{F}}(t), &(3.56)\end{aligned}$$

where $v^{\mathcal{F}}(s) = \tau^{\mathcal{F}}_{t,s} X_{c(s)}$. The proof is analogous to the one of Theorem 2.8.4.

- Following the previous comment and in analogy to the extension of the covariant derivative to tensor fields using parallel transport (cf. Secs. 2.8 and 2.9), one can first generalize Fermi transport to tensors to obtain an isomorphism

$$\begin{aligned} \tau^{\mathcal{F}}_{t,s} : T_{c(s)}M^r_s &\to T_{c(t)}M^r_s &(3.57)\\ T &\mapsto \tau^{\mathcal{F}}_{t,s} T, &(3.58)\end{aligned}$$

and then generalize the Fermi-derivative to tensor fields (making use of the identity (3.54)–(3.56); in analogy to Def. 2.9.1). The Fermi-derivative of tensor fields then has the following properties:

(i) $\mathcal{F}_u$ maps tensor fields onto tensor fields of the same rank (cf. Def. 2.9.1):

$$\mathcal{F}_u : \mathcal{T}^r_s M \rightarrow \mathcal{T}^r_s M \tag{3.59}$$
$$T \mapsto \mathcal{F}_u T \tag{3.60}$$

(ii) $\mathcal{F}_u f = u(f)$ if $f \in \mathcal{F}(M)$ (cf. Def. 2.9.1).

(iii) Product rule (cf. Theorem 2.9.2): Let $S \in \mathcal{T}^r_s(M)$, $T \in \mathcal{T}^p_q(M)$ be tensor fields of rank (r,s) and (p,q). Then:

$$\mathcal{F}_u(S \otimes T) = \mathcal{F}_u S \otimes T + S \otimes \mathcal{F}_u T. \tag{3.61}$$

(iv) $\mathcal{F}_u$ commutes with contractions as well as with raising and lowering indices (cf. Theorem 2.9.3).

Spin in Fermi frame. Let us consider a world line $c : I \subset \mathbb{R} \rightarrow M$ parametrized by proper time τ and assume we have an orthonormal frame $\{e_i(\tau)\}$, $i = 1,2,3$, along c, perpendicular to $e_0(\tau) = u(\tau) = \dot{c}(\tau)$. Noting that $g(u,u) = -1$, this means that $g(e_\mu, e_\nu) = \eta_{\mu\nu}$ along c. We can construct such a frame, e.g, by starting with such a local basis $\{e_\mu(\tau_0)\} \in T_{c(\tau_0)}M$ in $T_{c(\tau_0)}M$ at some point $p = c(\tau_0)$ and then parallel transport it along c.

The Fermi derivative of these basis vectors can be expanded in the same basis,

$$\mathcal{F}_u e_\mu(\tau) = \omega^\nu_\mu(\tau) e_\nu(\tau). \tag{3.62}$$

Note that the coefficients $\omega^\nu_\mu(\tau)$ measure the deviation from Fermi transport. It can be easily shown that these coefficients are non-vanishing only for purely spatial indices, and that they are anti-symmetric (cf. Straumann 2013); this means that we can write them as $\omega_{ij} = \epsilon_{ijk}\Omega^k$ for some vector Ω^k, with ϵ_{ijk} being the anti-symmetric Levi-Cività symbol. Now let us consider the spin $S = S^i e_i$ of an elementary particle or macroscopic body. Its evolution along c is then determined by (cf. Eq. (3.51))

$$0 = \mathcal{F}_u S = \frac{\mathrm{d}S^i}{\mathrm{d}\tau} e_i + S^j \mathcal{F}_u e_j = \frac{\mathrm{d}S^i}{\mathrm{d}\tau} e_i + S^j \omega^i_j e_i, \tag{3.63}$$

or,

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}\tau} = \mathbf{S} \times \mathbf{\Omega}. \tag{3.64}$$

If, however, $\{e_\mu(\tau)\}$ are Fermi-transported along c, then by construction $\Omega = 0$ (cf. Eq. (3.62)) and

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}\tau} = 0. \tag{3.65}$$

Therefore, a Fermi-transported frame is a special frame in which the spin of a particle or body does not precess. It is a non-rotating coordinate system for the observer with world line c.

Fermi normal coordinates. The construction of a Fermi-transported frame considered above gives rise to Fermi normal coordinates. Let $\{e_\mu(\tau)\}$ be a Fermi-transported orthonormal frame along c as constructed above, i.e.,

$$g(e_\mu(\tau), e_\nu(\tau)) = \eta_{\mu\nu}, \quad \mathcal{F}_u e_\mu(\tau) = 0. \tag{3.66}$$

Making use of the exponential map (cf. Def. 2.10.2), we can now define coordinates $x^\mu = (x^0, x^1, x^2, x^3)$ relative to this tetrad on c, the so-called Fermi-Walker coordinates:

$$x^0(p) = x^0(\exp_{c(\tau)}[\lambda^j e_j(\tau)]) = \tau, \quad x^i(p) = x^i(\exp_{c(\tau)}[\lambda^j e_j(\tau)]) = \lambda^i. \tag{3.67}$$

Here, we assume λ^i are sufficiently small (p sufficiently close to c) such that the exponential maps exist. One can show that there exists a neighborhood U of c on which this map

$$\phi : U \to \mathbb{R}^4 \tag{3.68}$$

$$p \mapsto \phi(p) = (x^0(p), x^1(p), x^2(p), x^3(p)) \tag{3.69}$$

is well defined and on which it is a diffeomorphism onto its image. Furthermore, one can show that the only non-vanishing Christoffel symbols in these coordinates are given by (Misner et al. 1973; Straumann 2013)

$$\Gamma^0_{i0} = \Gamma^i_{00} = a^i. \tag{3.70}$$

Therefore, we arrive at the important conclusion that if c is a geodesic, $a = \nabla_{\dot{c}}\dot{c} = 0$, and all Christoffel symbols vanish. This shows that Fermi transport generates **normal coordinates along an entire geodesic**. Furthermore, both terms on the right-hand side of Eq. (3.49) then vanish and we obtain $\mathrm{d}S^\mu/\mathrm{d}\tau = 0$, which we have already obtained above in a different way.

Chapter 4

Einstein's Field Equations

> "Dem Zauber dieser Theorie wird sich kaum jemand entziehen können, der sie wirklich erfaßt hat; sie bedeutet einen wahren Triumph der durch Gauss, Riemann, Christoffel, Ricci und Levi-Cività begründeten Methode des allgemeinen Differentialkalküls."
>
> "Nobody who really grasped it can escape from its charm, because it signifies a real triumph of the general differential calculus as founded by Gauss, Riemann, Christoffel, Ricci, and Levi-Civita."
>
> (A. Einstein, "On the general theory of relativity", Proceedings of the Royal Prussian Academy of Sciences (1915): 778-786. See also: The Collected Papers of Albert Einstein, Vol. 6, Doc. 21)

Einstein's field equations are postulated as an axiom in general relativity; thus they do not require a derivation or formal 'proof'. However, some of the ideas that led to this postulate are certainly of relevance. We shall first discuss the field equations from such a historical and heuristic point of view (Sec. 4.1). It was only realized much later (Lovelock 1972) that these field equations are unambiguous and uniquely defined, given certain well-motivated criteria. We shall comment on this result in Sec. 4.2. In Sec. 4.3, we impose the correspondence principle (cf. Sec. 3.1) and fix the coupling constant between spacetime curvature and the energy-momentum tensor in Einstein's equations. Finally, we discuss how to obtain the field equations from an action principle (Sec. 4.4), which is the starting point for alternative theories of gravity.

4.1 Heuristic motivation of the Field Equations

As a starting point, let us consider Einstein's idea that matter curves spacetime. Unlike in Newtonian mechanics, he did not think about gravity as a force, but rather as a property of space. Indeed, as already mentioned in Sec. 1.2, according to the WEP all bodies experience the same acceleration $g = GM/r^2$ regardless of their own mass. Therefore, one may indeed think of g as a property of space, rather than as a gravitational force in the Newtonian sense. In order to allow for spacetime to be curved, we must depart from Minkowskii space and consider four-dimensional Lorentzian manifolds. The equivalence principle is then embodied mathematically by the existence of local normal coordinates (see Sec. 3.1).

The description of matter, i.e., the energy and momentum distribution, is captured by the energy-momentum tensor, a rank-two tensor $T \in \mathcal{T}^0_2(M)$. We indeed already identified it as a source term in our attempt to generalize Newton's field equation to special relativity (cf. right-hand side of Eq. (1.19)). Therefore, curvature also needs to be described by a $\mathcal{T}^0_2(M)$ tensor field. One obvious candidate would be the Ricci tensor (cf. Def. 2.11.10), and a first guess for the field equations would be

$$\mathrm{Ric} = \kappa T, \tag{4.1}$$

where κ is a yet to be specified constant. This was indeed Einstein's first ansatz on November 4, 1915.[1] However, according to Theorem 2.11.11 this version of the field equations violates energy and momentum conservation: taking the covariant divergence of Eq. (4.1) one obtains the non-zero term $(1/2)gR_{;i}$ on the left-hand side. This problem can be avoided by postulating

$$G \equiv \mathrm{Ric} - \frac{1}{2}gR = \kappa T, \tag{4.2}$$

which is equivalent to the final form of the field equations Einstein presented to the Prussian Academy of Science on 25th November 1915.[2] The tensor $G \in \mathcal{T}^0_2(M)$ on the left-hand side is referred to as the Einstein tensor. The proportionality constant κ is determined by the correspondence principle (cf. Sec. 3.1), i.e., by requiring that these field equations recover Newton's field equations in the limit of "weak" gravitational fields. We shall determine this constant explicitly in Sec. 4.3. In components, Eq. (4.2) reads

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}. \tag{4.3}$$

We note that by contraction,

$$R^{\mu}{}_{\mu} - \frac{1}{2}\delta^{\mu}_{\mu}R = \kappa T^{\mu}{}_{\mu}, \tag{4.4}$$

one finds $R - (1/2)4R = -R = \kappa T$, where T denotes the contracted energy momentum tensor here ($T = T^{\mu}{}_{\mu}$). Substituting this into Eq. (4.2), we find

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right). \tag{4.5}$$

In the absence of matter, $T_{\mu\nu} = 0$, and this version of the field equations reduces to Eq. (4.1). Einstein has thus first found the correct vacuum field equations. Pseudo-Riemannian manifolds of this kind ($\mathrm{Ric} = 0$) are called Ricci-flat manifolds.

Written out in components in a given chart, Einstein's equations are highly non-linear partial differential equations in the metric components, even in vacuum. This is not surprising as according to the mass-energy equivalence any form of energy, even the energy associated with the gravitational field itself, corresponds to a mass and thus constitutes a source of the gravitational field. The absence of non-linearity was, in fact, a deficiency of the field equations that Einstein obtained in an attempt to generalize Newton's field equations within the framework of special relativity (cf. Sec. 1.2), and it is one reason why that attempt was doomed to fail.

[1] See Eq. (16) in A. Einstein, "On the general theory of relativity", Proceedings of the Royal Prussian Academy of Sciences (1915): 778-786. See also: The Collected Papers of Albert Einstein, Vol. 6, Doc. 21, https://einsteinpapers.press.princeton.edu/papers

[2] See Eq. (2a) in A. Einstein, "The Field Equations of Gravitation", Proceedings of the Royal Prussian Academy of Sciences (1915): 844-847. See also: The Collected Papers of Albert Einstein, Vol. 6, Doc. 25, https://einsteinpapers.press.princeton.edu/papers

On a historical note, Einstein got very excited about his first ansatz for the field equations (4.1), since using these equations he could explain the perihelion advance of Mercury. However, as he mentions in the introduction of his November 25 paper, the new (corrected) field equations still preserve this property. We will discuss the perihelion advance in Sec. 5.2.2.

Since $\nabla_X g = 0$ (see Exercise 2.9.7), another option for a divergence-less tensor on the left-hand side of the field equations would be $\mathrm{Ric} - \frac{1}{2} g R + \Lambda g$ for a so-called **cosmological constant** Λ, i.e.,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{4.6}$$

Indeed, Einstein preferred the latter equation with a small cosmological constant for a while, since soon after the 1915 breakthrough he realized that his original field equations (4.2) did not allow for static cosmological solutions.[3] The expansion of the universe was discovered by Hubble only in 1929, and prior to that time a dynamical, time-evolving universe seemed rather absurd.

Exercise 4.1.1. ***(deSitter spacetime)***

(a) *Show that Einstein's field equations in vacuum with a cosmological constant can be written as*

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{4.7}$$

(b) *Consider the metric*

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \exp(2\alpha t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{4.8}$$

and show that it is a solution of Eq. (4.7) for $\Lambda > 0$ with $\alpha = \sqrt{\Lambda/3}$. This is a portion of the so-called deSitter spacetime.

4.2 Uniqueness of field equations

The field equations (4.6) had been 'derived' heuristically, and so it remained an open question whether these equations are ambiguous or uniquely determined in some sense; that is, are there potential other field equations that would be consistent with both the equivalence principle and observational tests? Some of the practical consequences of Einstein's field equations have been worked out early on, such as the perihelion advance of Mercury. Einstein derived confidence in the validity of his field equations from the fact that their predictions appeared to be accurate.

It has been realized only much later that the field equations are uniquely determined under the following two conditions:

1) Since the Riemann curvature tensor can be constructed entirely from the metric components $g_{\mu\nu}$ and their first and second derivatives (cf. Theorem 2.11.3), the left-hand side of the field equations describing curvature must be a function thereof. We thus have the ansatz:

$$\mathcal{G}_{\mu\nu}[g_{\mu\nu}, \partial_\mu g_{\alpha\beta}, \partial^2 g_{\alpha\beta}] = T_{\mu\nu}, \tag{4.9}$$

where $\mathcal{G} \in \mathcal{T}_2^0(M)$ is a smooth tensor field constructed locally from $g_{\mu\nu}$ as well as the first and second derivatives.

[3] The cosmological constant is introduced in Eq. (13a) of A. Einstein "Cosmological Considerations in the General Theory of Relativity", Proceedings of the Royal Prussian Academy of Sciences (1917): 142-152. See also: The Collected Papers of Albert Einstein, Vol. 6, Doc. 43, https://einsteinpapers.press.princeton.edu/papers

2) In order to respect energy and momentum conservation, one must require

$$\nabla_\mu \mathcal{G}^{\mu\nu}[g_{\mu\nu}, \partial_\mu g_{\alpha\beta}, \partial^2 g_{\alpha\beta}] = 0. \tag{4.10}$$

Theorem 4.2.1. ***(Lovelock 1972)*** *A smooth tensor field $\mathcal{G} \in \mathcal{T}^0_2(M)$ on a four-dimensional (pseudo-)Riemannian manifold (M, g) that satisfies the properties 1) and 2) is a linear combination of the Einstein tensor and the metric tensor ($a, b \in \mathbb{R}$):*

$$\mathcal{G}_{\mu\nu}[g_{\mu\nu}] = aG_{\mu\nu} + bg_{\mu\nu}. \tag{4.11}$$

We shall not prove this theorem here, and instead refer to Lovelock (1972) (or to Straumann 2013 for a weaker version of it). The proof is well readable, but too lengthy to be reproduced here.

4.3 The Newtonian limit

Einstein's field equations (4.2) contain the proportionality constant κ, which describes the coupling between matter and curvature. We shall now determine it by imposing the correspondence principle (cf. Sec. 3.1), i.e., by requiring that the field equations reduce to Newton's field equations in the limit of 'weak' stationary gravitational fields (the non-relativistic limit). In order to do so, we shall first show that in the weak-field limit, the geodesic equations reduce to the Newtonian equation of motion in a gravitational field. Using this result we will then show that Einstein's field equations reduce to the Newtonian field equations and determine κ.

For weak gravitational fields, one would expect that nearly Lorentzian systems exists, i.e., that there should exist local charts (U, ϕ) with coordinates (x^0, x^1, x^2, x^3) in which the metric components can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1. \tag{4.12}$$

Newton's equation of motion. Consider a slowly moving particle on its geodesic (Eq. (3.11)) parametrized by proper time (Eq. (3.12)). We then have $v^i \equiv \mathrm{d}x^i/\mathrm{d}\tau \ll 1$. We can thus neglect terms containing $\mathrm{d}x^i/\mathrm{d}\tau$ with respect to terms containing $\mathrm{d}x^0/\mathrm{d}\tau$ in the geodesic equation and the normalization condition (3.12), since the latter equation, taken together with the metric components (4.12), implies $\mathrm{d}x^0/\mathrm{d}\tau \simeq 1$. Therefore,

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = -\Gamma^\mu_{\nu\delta}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\delta}{\mathrm{d}\tau} \simeq -\Gamma^\mu_{00}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^2. \tag{4.13}$$

For quasi-stationary fields we can neglect time derivatives and obtain to first order in $h_{\mu\nu}$ (cf. Eq. (2.135))

$$\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu i}\partial_i g_{00} \simeq -\frac{1}{2}\eta^{\mu i}\partial_i h_{00} = \begin{cases} 0, & \mu = 0 \\ -\frac{1}{2}\partial_i h_{00}, & \mu = i \in \{1,2,3\} \end{cases}. \tag{4.14}$$

We thus find $\mathrm{d}^2t/\mathrm{d}\tau^2 \equiv \mathrm{d}^2x^0/\mathrm{d}\tau^2 = 0$, i.e., $\mathrm{d}t/\mathrm{d}\tau = \text{const.}$, and conclude that to first order in $h_{\mu\nu}$

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} \simeq \frac{1}{2}\partial_i h_{00}. \tag{4.15}$$

This is equivalent to the Newtonian equation of motion (cf. Eq. (1.5)),

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\nabla\Phi, \tag{4.16}$$

provided we identify h_{00} with the Newtonian gravitational potential Φ via

$$h_{00} \simeq -2\Phi + \text{const.} \tag{4.17}$$

For isolated systems, Φ and h_{00} must vanish at infinity, which implies (in geometric units, $\hbar = c = 1$):

$$g_{00} \simeq -1 - 2\Phi, \quad |\Phi| \ll 1. \tag{4.18}$$

This Newtonian limit is well justified for a number of objects. We estimate the surface strength of the Newtonian gravitational potential for a number of objects,

$$\Phi_0 = -\frac{GM}{R_0}, \tag{4.19}$$

where R_0 is the surface radius, and find:

$$\left|\frac{\Phi_0}{c^2}\right| \approx \begin{cases} 10^{-39}, & \text{proton} \\ 1\times 10^{-9}, & \text{Earth} \\ 2\times 10^{-6}, & \text{Sun} \\ 1.5\times 10^{-4}, & \text{white dwarf} \\ 1.5\times 10^{-1}, & \text{neutron star} \end{cases}. \tag{4.20}$$

Note, however, that we have not obtained information on the other components of $h_{\mu\nu}$ yet, and that they must not be necessarily small compared to h_{00}. Einstein was neglecting this fact in 1911 when he computed the deflection of light and obtained an angle that is only half of the correct GR result[4] (see also Secs. 5.2.3 and 7.2).

It is obvious from Eqs. (4.18) and (4.19) that the correction to flat spacetime is determined by the **compactness** of an object, i.e., by the quotient M/R of an object. While the Newtonian limit appears to be well justified for ordinary stars such as the Sun, the estimates in Eq. (4.20) indicate that there should be non-negligle corrections for more compact objects such as neutron stars, which we will discuss in Chap. 6. For the even more compact black holes, relativistic effects are so large that they cannot be treated as corrections anymore (Sec. 5.3).

Newton's field equation. The components of the Ricci tensor (cf. Eq. (2.286) and Def. 2.11.10) in the stationary, weak-field limit up to first order in $h_{\mu\nu}$ can be written as

$$R_{\mu\nu} \simeq \partial_l \Gamma^l_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\lambda\mu}. \tag{4.21}$$

Here, we have ignored all terms quadratic in the Christoffel symbols, as they are second-order in $h_{\mu\nu}$ (cf. Eq. (2.135)). Therefore, we find using Eqs. (4.14) and (4.17)

$$R_{00} \simeq \partial_l \Gamma^l_{00} \simeq -\frac{1}{2}\partial^l \partial_l h_{00} \simeq \Delta\Phi \tag{4.22}$$

[4] A. Einstein, "On the Influence of Gravitation on the Propagation of Light", Annalen der Physik 35 (1911), 898-908. See also: The Collected Papers of Albert Einstein, Vol. 3, Doc. 23, https://einsteinpapers.press.princeton.edu/papers

For non-relativistic velocities, $|T_{ij}|, |T_{0j}| \ll |T_{00}|$, as T_{0j} are linear and T_{ij} are quadratic in the velocities (cf. Eq. (1.17)). According to Einstein's equations we then also have $|G_{ij}|, |G_{0j}| \ll |G_{00}|$. Computing the trace of the Einstein tensor, we obtain the exact expression

$$g^{\mu\nu}G_{\mu\nu} = R - 2R = -R, \tag{4.23}$$

and an approximate expression up to first order in $h_{\mu\nu}$ (note that R and $R_{\mu\nu}$ are at least first order in $h_{\mu\nu}$):

$$g^{\mu\nu}G_{\mu\nu} \approx g^{00}G_{00} = \eta^{00}R_{00} - \frac{1}{2}\eta^{00}\eta_{00}R = -R_{00} - \frac{1}{2}R. \tag{4.24}$$

The previous two expressions imply $R = 2R_{00}$, and thus we find up to first order in $h_{\mu\nu}$:

$$G_{00} = R_{00} - \frac{1}{2}\eta_{00}R = 2R_{00} = 2\Delta\Phi, \tag{4.25}$$

where we have also used Eq. (4.22). Recalling that $T_{00} = \rho$ (cf. Eq. (1.17)) we conclude that the 00-component of Einstein's equations,

$$G_{00} = \kappa T_{00}, \tag{4.26}$$

reduces to Newton's field equation (cf. Eq. (1.7)),

$$\Delta\Phi = 4\pi G\rho, \tag{4.27}$$

if we identify

$$\kappa = 8\pi G. \tag{4.28}$$

This identification fixes the final form of Einstein's field equations. Finally, we note that the Newtonian limit of Einstein's field equations with a cosmological constant (Eq. (4.6)) is

$$\Delta\Phi = 4\pi G\rho + \Lambda, \tag{4.29}$$

where we have assumed that Λ is small enough such that we can neglect the term $h_{00}\Lambda$.

4.4 Lagrangian formulation

In this section, we shall show that as in the case of the equation of motion (the geodesic equations; cf. Exercise 2.10.6) Einstein's field equations can be obtained from a variational principle based on the so-called Einstein-Hilbert action. The significance of this is that such an action principle provides a natural starting point for deriving field equations for alternative theories of gravity. We shall explore the particular example of so-called $f(R)$-theories in an exercise below.

Theorem 4.4.1. *Let (M, g) be a spacetime. Einstein's field equations with a cosmological constant Λ (Eq. (4.6)) can be obtained from the variational principle*

$$\delta_g \int_D (\mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{mat}})\text{dvol}_g = 0, \tag{4.30}$$

where $D \subset M$ is a compact region in spacetime with boundary ∂D,

$$\mathcal{L}_{\text{grav}} = \frac{1}{16\pi G}(R - 2\Lambda) \tag{4.31}$$

*is the **Einstein-Hilbert action**, and*

$$\mathcal{L}_{\text{mat}} = \mathcal{L}_{\text{mat}}(\Psi_I, \nabla\Psi_I, g) \tag{4.32}$$

is the Lagrangian density for n matter fields Ψ_I, with $I = \{1, \ldots, n\}$. Furthermore, δ_g denotes the variational derivative $\partial/\partial\epsilon|_{\epsilon=0}$ with respect to a one-parameter family of metrics g_ϵ with $\epsilon \in (-\alpha, \alpha)$.

Proof. For simplicity, we assume that D is entirely contained within the domain of a chart (U, ϕ) of M. The final result, however, holds independently of this assumption.

First, let us calculate the variational derivative of the Riemann volume element $\mathrm{dvol}_g = \sqrt{-g}\mathrm{d}^4x$. We start by noting that

$$0 = \delta_g(\delta^\mu_\nu) = \delta_g(g^{\mu\lambda}g_{\nu\lambda}) = g_{\nu\lambda}\delta_g g^{\mu\lambda} + g^{\mu\lambda}\delta_g g_{\nu\lambda} \quad \Rightarrow \quad g^{\mu\lambda}\delta_g g_{\nu\lambda} = -g_{\nu\lambda}\delta_g g^{\mu\lambda}. \tag{4.33}$$

Multiplying by $g_{\alpha\mu}$ and summing over μ, one also finds:

$$\delta_g g_{\nu\alpha} = -g_{\alpha\mu}g_{\nu\lambda}\delta_g g^{\mu\lambda}. \tag{4.34}$$

Using Eq. (4.33) and Cramer's rule to differentiate the determinant of the metric g, we find:

$$\delta_g\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta_g g = -\frac{1}{2\sqrt{-g}}g g^{\mu\nu}\delta_g g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_g g^{\mu\nu}. \tag{4.35}$$

Therefore,

$$\delta_g \mathrm{dvol}_g = -\frac{1}{2}g_{\mu\nu}\delta_g g^{\mu\nu}\,\mathrm{dvol}_g, \tag{4.36}$$

and for the cosmological constant term in $\mathcal{L}_{\text{grav}}$ we immediately obtain

$$\delta_g\int_D \frac{-2\Lambda}{16\pi G}\mathrm{dvol}_g = \int_D \frac{1}{16\pi G}\Lambda g_{\mu\nu}\delta_g g^{\mu\nu}\,\mathrm{dvol}_g. \tag{4.37}$$

For the curvature term we calculate:

$$\begin{aligned}
\delta_g\int_D R\,\mathrm{dvol}_g &= \int_D \delta_g(g^{\mu\nu}R_{\mu\nu}\sqrt{-g})\mathrm{d}^4x && (4.38)\\
&= \int_D R_{\mu\nu}\delta_g g^{\mu\nu}\,\mathrm{dvol}_g + \int_D R\,\delta_g\mathrm{dvol}_g + \int_D g^{\mu\nu}\delta_g R_{\mu\nu}\,\mathrm{dvol}_g && (4.39)\\
&= \int_D G_{\mu\nu}\delta_g g^{\mu\nu}\,\mathrm{dvol}_g + \int_D g^{\mu\nu}\delta_g R_{\mu\nu}\,\mathrm{dvol}_g, && (4.40)
\end{aligned}$$

where we have again used the identity (4.36) in the last step. We will now show that the last term on the right-hand side can be written as the divergence of a vector field, which vanishes at the boundary ∂D if the variations $\delta_g g_{\mu\nu}$ vanish outside of D. According Gauss' theorem for integration on manifolds, this term then vanishes. For simplicity, let us evaluate $\delta_g R_{\mu\nu}$ at $p \in M$ in normal coordinates. Making use that partial derivatives commute, we have (cf. Eq. (2.286) and Def. 2.11.10)

$$\delta_g R_{\mu\nu} = \partial_\lambda(\delta_g\Gamma^\lambda_{\mu\nu}) - \partial_\nu(\delta_g\Gamma^\lambda_{\mu\lambda}). \tag{4.41}$$

From the transformation property of the Christoffel symbols (Eq. (2.136)),

$$\delta_g\bar{\Gamma}^i_{jk} = \frac{\partial x^l}{\partial\bar{x}^j}\frac{\partial x^m}{\partial\bar{x}^k}\frac{\partial\bar{x}^i}{\partial x^n}\delta_g\Gamma^n_{lm}, \tag{4.42}$$

i.e., the variational derivative of the Christoffel symbols transforms as a tensor under a change of coordinates. Therefore, at p one can write

$$\delta_g R_{\mu\nu} = \nabla_\lambda(\delta_g \Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta_g \Gamma^\lambda_{\mu\lambda}), \tag{4.43}$$

which is known as the Palatini identity. Since it is a tensor equation it holds in any chart. Therefore, using the fact that $\nabla_X g \equiv 0$ (cf. Exercise 2.9.7) one finds:

$$\begin{aligned} g^{\mu\nu}\delta_g R_{\mu\nu} &= \nabla_\lambda(g^{\mu\nu}\delta_g \Gamma^\lambda_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta_g \Gamma^\lambda_{\mu\lambda}) && (4.44)\\ &= \nabla_\lambda(g^{\mu\nu}\delta_g \Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\delta_g \Gamma^\nu_{\mu\nu}) && (4.45)\\ &\equiv \nabla_\lambda V^\lambda, && (4.46) \end{aligned}$$

where we have defined a vector field V. Using Gauss' theorem, we thus find that the last term on the right-hand side of Eq. (4.40) can be written as

$$\int_{\partial D} V^\lambda n_\lambda \, \mathrm{d}A_g = 0, \tag{4.47}$$

where n^λ is the unit normal on ∂D and $\mathrm{d}A_g$ is the surface element. Here, we assumed that all metric variations are confined to within D (i.e., that V vanishes on ∂D).

The variational derivative of the matter part is given by

$$\begin{aligned} \delta_g \int_D \mathcal{L}_{\mathrm{mat}} \, \mathrm{dvol}_g &= \int_D \delta_g \mathcal{L}_{\mathrm{mat}} \, \mathrm{dvol}_g + \int_D \mathcal{L}_{\mathrm{mat}} \delta_g \mathrm{dvol}_g && (4.48)\\ &= \int_D (\delta_g \mathcal{L}_{\mathrm{mat}} + \frac{1}{2}\mathcal{L}_{\mathrm{mat}} g^{\mu\nu}\delta_g g_{\mu\nu}) \, \mathrm{dvol}_g && (4.49)\\ &= \frac{1}{2}\int_D T^{\mu\nu}\delta_g g_{\mu\nu} \, \mathrm{dvol}_g && (4.50)\\ &= -\frac{1}{2}\int_D T_{\mu\nu}\delta_g g^{\mu\nu} \, \mathrm{dvol}_g, && (4.51) \end{aligned}$$

where we have again made use of the identities (4.36), (4.33), and (4.34). Furthermore, we have employed the general definition of the energy-momentum tensor in a Lagrangian field theory (which is symmetric by construction).

Reassembling Eqs. (4.37), (4.40), and (4.51), we find that the variational derivative Eq. (4.30) is given by

$$\int_D \left[\frac{1}{16\pi G}(G_{\mu\nu} + \Lambda g_{\mu\nu}) - \frac{1}{2}T_{\mu\nu}\right] \delta_g g^{\mu\nu} \, \mathrm{dvol}_g = 0. \tag{4.52}$$

For this to hold true for any one-parameter family of metrics, the expression in square brackets must vanish, which is equivalent to Einstein's field equations with a cosmological constant (4.6). □

Some **remarks**:

- The matter fields in $\mathcal{L}_{\mathrm{mat}}$ in the previous theorem are assumed to be tensor fields, such as, e.g., the electromagnetic fields and/or the matter fields of an ideal fluid. Usually, $\mathcal{L}_{\mathrm{mat}}$ is known in special relativity. The general-relativistic form of $\mathcal{L}_{\mathrm{mat}}$ required in the previous theorem can then be obtained with the equivalence principle by following the substitution rules (see Sec. 3.1, Eqs. (3.3)–(3.5)).

- In the action (4.30), contributions to the Lagragian from the gravitational field and from the matter fields appear as two separate terms. The gravitational Lagrangian $\mathcal{L}_{\text{grav}}$ defines the contributing fields ($g_{\mu\nu}$, possibly additional fields in alternative theories) and describes the self-interaction (non-linearities of the theory) as well as the dynamics of gravity. The matter Lagrangian $\mathcal{L}_{\text{mat}}$ describes all non-gravitational fields and is usually a generalization of a special-relativistic theory (see comment above). As a result of the equivalence principle and the substitution rules, $\mathcal{L}_{\text{mat}}$ depends on the metric either explicitly or implicitly (through covariant derivatives). The coupling between matter fields and the gravitational field is thus achieved by $g_{\mu\nu}$ in $\mathcal{L}_{\text{mat}}$.

- An action of the form (4.30) is usually the **starting point for alternative theories of gravity**, since such a variational principle provides a natural way to obtain alternative field equations. Typically, $\mathcal{L}_{\text{grav}}$ is replaced by an alternative ansatz. If an alternative theory introduces new fields, a good way to do so is through $\mathcal{L}_{\text{grav}}$ rather than $\mathcal{L}_{\text{mat}}$; this makes it easier to avoid a violation of the equivalence principle, which has been verified to high precison (see Sec. 1.2). Many current attempts to extend theoretical physics (string theory, supersymmetric theories, etc.) introduce new fields, which contribute to gravity. They change the dynamics of the gravitational field $\mathcal{L}_{\text{grav}}$, but often also the coupling of matter to the gravitational field(s). However, for some theories the violation of the equivalence principle is so small that they are still consistent with experimental tests of the equivalence principle.

Exercise 4.4.2. ***(Energy momentum tensor in Lagrangian field theories)***
In this problem, we consider electrodynamics as an example for a Lagrangian field theory and obtain the energy momentum tensor from the general expression Eq. (4.49). The Lagrangian for the electromagnetic field is given by

$$\mathcal{L}_{\text{EM}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}, \tag{4.53}$$

where $F_{\mu\nu}$ is the antisymmetric electromagnetic field tensor. From Eq. (4.49) derive the well-known result

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha} F_{\nu\beta} g^{\alpha\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right). \tag{4.54}$$

Exercise 4.4.3. ***(Action principle: alternative theories of gravity)***
Follow the proof of Theorem 4.4.1 and generalize it to the case in which the Einstein-Hilbert action is replaced by

$$\mathcal{L}_{\text{grav}} = f(R), \tag{4.55}$$

where f is a function of the curvature scalar R. The result is:

$$\delta_g \int_D f(R)\,\text{dvol}_g = \int_D \left[R_{\mu\nu} f'(R) - \frac{1}{2} g_{\mu\nu} f(R) + g_{\mu\nu} \nabla^2 f'(R) - \nabla_\mu \nabla_\nu f'(R) \right] \delta_g g^{\mu\nu}\,\text{dvol}_g. \tag{4.56}$$

These so-called "$f(R)$-theories" are a popular class of alternative theories of gravity.

Chapter 5

The Schwarzschild solution

5.1 Isotropic solution of Einstein's Field Equations

In this section, we shall derive the most simple non-trivial solution to Einstein's equations in three spatial dimensions, an isotropic solution. We will derive both the outer (vacuum) solution that applies to the exterior of an isotropic matter distribution, as well as the interior (matter) solution inside the matter distribution itself. The latter component already provides a starting point for computing the structure of relativistic stars (Chap. 6).

The vacuum solution was first derived by Karl Schwarzschild—then professor at the Potsdam Observatory—only a few months after Einstein had published his field equations.[1] He did so after having returned from World War I with serious illness in the fall of 1915—a few months before his death. Schwarzschild did not only find this solution, he also derived the perihelion advance of Mercury and the deflection of light in his metric, which Einstein had only calculated in the post-Newtonian approximation until then. Only one month after presenting the vacuum solution, Schwarzschild also published the non-vacuum solution for a spherical distribution of incompressible matter of finite radius R.[2]

Proposition 5.1.1. ***(Schwarzschild solution)***
There exists an isotropic static solution to the Einstein equations. We assume an isotropic matter distribution (ideal fluid) inside a spatial region with radius R. One can write this solution as the Lorentzian manifold (M, g) with $M = \mathbb{R} \times (0, \infty) \times S^2$, coordinates (t, r, θ, ϕ), where θ and ϕ are the standard spherical coordinates of the unit sphere $S^2 \subset \mathbb{R}^3$, and metric

$$(g_{ij}) \quad = \quad \mathrm{diag}\left(-\left(1-\frac{2Gm}{r}\right), \left(1-\frac{2Gm}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right), \quad r > R, \tag{5.1}$$

$$(g_{ij}) \quad = \quad \mathrm{diag}\left(A(r), \left(1-\frac{2Gm(r)}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right), \quad r < R, \tag{5.2}$$

[1] K. Schwarzschild, "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie", Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Proceedings of the Prussian Academy of Sciences) **7**, 189, January 13, 1916.

[2] K. Schwarzschild, "Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie", Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Proceedings of the Prussian Academy of Sciences) **7**, 424, February 24, 1916.

where

$$A(r) = \exp\left[-\int_r^\infty \left(8\pi G p(s) s + \frac{2Gm(s)}{s}\right)\left(1 - \frac{2Gm(s)}{s}\right)^{-1} \mathrm{d}s\right], \tag{5.3}$$

with p and ρ being the pressure and the <u>*total*</u> *mass-energy density*[3] *of the matter distribution, $m = m(R)$ the total gravitational mass of mass distribution $m(r)$, and G denotes the gravitational constant. The exterior solution is necessarily static for $r > 2Gm$.*

Let us preface the derivation of this isotropic static **Schwarzschild solution** with a few comments:

1. The fact that isotropy (spherical symmetry) necessarily implies a static solution (for $r > 2Gm$) is known as the **Birkhoff theorem**.

2. In the vacuum solution, $r > R$ (Eq. (5.1)), m is an integration constant and thus a free parameter of the solution. We shall later see that this can be interpreted as the total **gravitational mass** of the matter distribution. Since this solution applies to the outside of any isotropic matter distribution, we shall see that it describes spacetime around (non-rotating) astronomical objects, such as stars.

3. The Schwarzschild solution is not defined at $r = 2Gm$, the so-called **Schwarzschild radius**,
$$R_\mathrm{S} = 2\frac{Gm}{c^2} \tag{5.4}$$
(in SI-units). In principle, we must exclude the sphere of radius R_S from the spacetime and would thus get two disconnected spacetime components corresponding to $r \in (0, 2Gm)$ and $r \in (2Gm, \infty)$. However, we shall later see that the divergence of the metric components at $r = 2Gm$ is only the result of a coordinate singularity, i.e., it is due to a bad choice of coordinates at the Schwarzschild radius. This divergence, however, does not represent a spacetime singularity, and one can find continuations of coordinates across $r = 2Gm$ (see, e.g., Sec. 5.5).

4. The exterior solution $r > R$ does not make any reference to the spatial distribution of matter, i.e., the matter distribution can be thought of as a point-like object of mass m located at $r = 0$. This is analogous the the electric field of a spherical charge distribution.

5. For typical astronomical objects such as normal stars, $R_\mathrm{S} = 2Gm$ is tiny ($\sim$few km for the Sun, several mm for the Earth). However, for very compact objects, R may be smaller than $2Gm$; we shall identify such objects as **black holes** (Sec. 5.3).

Proof. The requirement of isotropy implies that M must have two-dimensional spacelike surfaces that are spherically symmetric. This is satisfied by construction if we choose M to be a family of spheres, parametrized by some radius $r \in (0, \infty)$ and time coordinate $t \in \mathbb{R}$: $M = \mathbb{R} \times (0, \infty) \times S^2$. Every sphere can be identified with the unit sphere S^2 by adopting the usual spherical polar coordinates θ and ϕ. This means that the metric g restricted to the two-dimensional tangent spaces spanned by ∂_θ and ∂_ϕ is the usual metric of spherical polar coordinates, i.e.,

$$g(\partial_\theta, \partial_\theta) = r^2, \quad g(\partial_\phi, \partial_\phi) = r^2 \sin^2\theta, \quad g(\partial_\phi, \partial_\theta) = 0. \tag{5.5}$$

[3] We shall use ρ here as an abbreviation for the rest-mass density ρ_b plus the internal energy density ϵ, $\rho \equiv \rho_\mathrm{b} + \epsilon$.

Because of spherical symmetry the angular distance between two points, e.g., $(t, r, \theta \pm \mathrm{d}\theta, \phi)$ and (t, r, θ, ϕ), cannot depend on the sign of $\pm\mathrm{d}\theta$; therefore, terms in the line element $\mathrm{d}s^2$ linear in $\mathrm{d}\phi$ or $\mathrm{d}\theta$ must not exist, which meant that

$$g(\partial_t, \partial_\theta) = g(\partial_t, \partial_\phi) = g(\partial_r, \partial_\theta) = g(\partial_r, \partial_\phi) = 0. \tag{5.6}$$

Furthermore, all remaining coefficients in the line element must not depend on ϕ and θ. We are thus led to consider a metric of the form

$$(g_{ij}) = \begin{pmatrix} -a(r,t) & c(r,t) & 0 & 0 \\ c(r,t) & b(r,t) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}. \tag{5.7}$$

Consider now the following coordinate transformation $t \to \bar{t} = t + \alpha(r,t)$, with a yet to be specified function $\alpha(r,t)$, and all other coordinates unchanged ($\bar{r} = r, \bar{\theta} = \theta, \bar{\phi} = \phi$). Employing the transformation rule (2.35), we find

$$\bar{\partial}_{\bar{r}} = -\alpha'(r,t)\partial_t + \partial_r, \quad \bar{\partial}_{\bar{t}} = \frac{1}{1+\dot{\alpha}(r,t)}\partial_t, \quad \bar{\partial}_{\bar{\theta}} = \partial_\theta \quad , \bar{\partial}_{\bar{\phi}} = \partial_\phi, \tag{5.8}$$

where α' and $\dot{\alpha}$ denote differentiation of α with respect to r and t, respectively. Setting

$$0 = g(\bar{\partial}_{\bar{t}}, \bar{\partial}_{\bar{r}}) = \frac{1}{1+\dot{\alpha}(r,t)} g(\partial_t, -\alpha'(r,t)\partial_t + \partial_r), \tag{5.9}$$

we obtain

$$\alpha'(r,t) = -\frac{c(r,t)}{a(r,t)}. \tag{5.10}$$

For given t, this equation can be integrated in r to find $\alpha(r,t)$. In these new coordinates, dropping all $^-$-signs, we thus have

$$(g_{\mu\nu}) = \mathrm{diag}(-A(r,t), B(r,t), r^2, r^2\sin^2\theta), \tag{5.11}$$
$$(g^{\mu\nu}) = \mathrm{diag}(-A^{-1}(r,t), B^{-1}(r,t), r^{-2}, r^{-2}\sin^{-2}\theta) \tag{5.12}$$

Using Eq. (2.135), one finds that the non-zero Christoffel Symbols for this metric are given by

$$\Gamma^0_{00} = \frac{\dot{A}}{2A}, \qquad \Gamma^1_{01} = \frac{\dot{B}}{2B}, \quad \Gamma^0_{11} = \frac{\dot{B}}{2A}, \tag{5.13}$$
$$\Gamma^1_{00} = \frac{A'}{2B}, \quad \Gamma^1_{11} = \frac{B'}{2B}, \quad \Gamma^1_{22} = -\frac{r}{B}, \quad \Gamma^1_{33} = -\frac{r\sin^2\theta}{B}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \tag{5.14}$$
$$\Gamma^0_{01} = \frac{A'}{2A}, \quad \Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^3_{32} = \cot\theta. \tag{5.15}$$

With the help of Eq. (2.286) and Def. 2.11.10 one computes the components of the Ricci tensor:

$$
\begin{aligned}
R_{00} &= \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{Br} - \frac{\ddot{B}}{2B} + \frac{\dot{B}}{4B}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right), && (5.16)\\
R_{01} &= \frac{\dot{B}}{Br}, && (5.17)\\
R_{11} &= \frac{\ddot{B} - A''}{2A} + \frac{(A')^2 - \dot{A}\dot{B}}{4A^2} + \frac{A'B' - (\dot{B})^2}{4AB} + \frac{B'}{Br}, && (5.18)\\
R_{22} &= \frac{r}{2B}\left(\frac{B'}{B} - \frac{A'}{A}\right) + 1 - \frac{1}{B}, && (5.19)\\
R_{33} &= \sin^2\theta R_{22}. && (5.20)
\end{aligned}
$$

Let us now consider the two regimes $r > R$ and $r < R$.

Vacuum solution (exterior Schwarzschild metric). For $r > R$ the energy-momentum tensor vanishes and Einstein's field equations reduce to $R_{\mu\nu} = 0$ (cf. Eq. (4.5)). From Eq. (5.17) we then immediately find

$$\dot{B}(r, t) = 0, \tag{5.21}$$

i.e., B is time-independent, and all terms with time derivatives in Eqs. (5.16)–(5.20) vanish. Furthermore, using $R_{00} = R_{11} = 0$, we then find

$$0 = \frac{R_{00}}{A} + \frac{R_{11}}{B} = \frac{A'}{ABr} + \frac{B'}{B^2 r} = \frac{1}{rB}\left(\frac{A'}{A} + \frac{B'}{B}\right) = \frac{1}{rB}(\log(AB))'. \tag{5.22}$$

Consequently, $AB =$ const., and A must be time independent as well. At infinity ($r \to \infty$) it is plausible to impose $\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1$, as we expect the metric to approach the Minkowski metric. Therefore, $B = 1/A$, and $R_{22} = 0$ becomes

$$0 = -\frac{A'r}{2} - \frac{A'r}{2} + 1 - A \quad \Leftrightarrow \quad (rA(r))' = 1. \tag{5.23}$$

Naming the integration constant $-2Gm$, we arrive at $rA(r) = r - 2Gm$ and the metric reads

$$(g_{\mu\nu}) = \mathrm{diag}\left(-\left(1 - \frac{2Gm}{r}\right), \left(1 - \frac{2Gm}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right). \tag{5.24}$$

This metric changes the timelike behaviour of the time coordinate for $r < 2Gm$, which becomes spacelike (cf. Sec. 5.3); therefore, although the coordinate t does not appear explicitly in the metric components, the solution is not static for $r < 2Gm$.

Matter solution (interior Schwarzschild metric). The energy momentum tensor of an ideal fluid,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{5.25}$$

reduces to

$$
\begin{aligned}
(T^{\mu\nu}) &= \mathrm{diag}(\rho A^{-1}, pB^{-1}, pr^{-2}, pr^{-2}\sin^{-2}\theta),, && (5.26)\\
(T_{\mu\nu}) &= \mathrm{diag}(\rho A, pB, pr^2, pr^2\sin^2\theta), && (5.27)
\end{aligned}
$$

if the fluid is at rest. Here, we have made use of the fact that $u^i = 0$ and (cf. Eq. (3.12)) $-1 = g(u,u) = -Au^0u^0$, i.e., $u^0 = A^{-1/2}$, and thus $u_0 = -Au^0 = -A^{1/2}$. The contraction of the energy-momentum tensor then becomes

$$T = T^{\mu}{}_{\mu} = g_{\mu\nu}T^{\mu\nu} = -\rho + 3p. \tag{5.28}$$

Therefore the right-hand side of Einstein's equations in the form (4.5) is given by

$$8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) = 4\pi G\,\mathrm{diag}((\rho+3p)A, (\rho-p)B, (\rho-p)r^2, (\rho-p)r^2\sin^2\theta). \tag{5.29}$$

Einstein's equations for R_{01} is thus identical to the vacuum case and we immediately conclude that $B(r,t) = B(r)$ does not depend on t. Einstein's equations for R_{00}, R_{11}, and R_{22} then read

$$\frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{Br} = 4\pi G(\rho+3p)A, \tag{5.30}$$

$$-\frac{A''}{2A} + \frac{(A')^2}{4A^2} + \frac{A'B'}{4AB} + \frac{B'}{Br} = 4\pi G(\rho-p)B, \tag{5.31}$$

$$\frac{r}{2B}\left(\frac{B'}{B} - \frac{A'}{A}\right) + 1 - \frac{1}{B} = 4\pi G(\rho-p)r^2. \tag{5.32}$$

Furthermore, we shall look for solutions with $A(r,t) = A(r)$. Multiplying these equations by $r^2/(2A)$, $r^2/(2B)$ and 1, respectively, and adding them yields

$$\frac{B'r}{B^2} + 1 - \frac{1}{B} = 8\pi G\rho r^2 \quad \Leftrightarrow \quad 1 - 8\pi G\rho r^2 = \left(\frac{r}{B}\right)'. \tag{5.33}$$

Integrating this equation, we obtain

$$\frac{r}{B(r)} = \int_0^r \left[1 - 8\pi G\rho(s)s^2\right]\mathrm{d}s \equiv r - 2Gm(r), \tag{5.34}$$

where

$$m(r) = \int_0^r 4\pi\rho(s)s^2\mathrm{d}s, \tag{5.35}$$

and thus

$$B(r) = \left(1 - \frac{2Gm(r)}{r}\right)^{-1}. \tag{5.36}$$

For $r > R$, $m(r) = m(R)$, and thus $B(r) = B(R)$. This means that if we choose $m = m(R)$ for the vacuum solution, $B(r)$ is also the coefficient of the vacuum solution outside the matter distribution and B is continuous at $r = R$.

In order to obtain $A(r)$, we substitute $B(r)$ into Eq. (5.32) and find:

$$\frac{A'}{A} = \left(8\pi Gp(r)r + \frac{2Gm(r)}{r^2}\right)\left(1 - \frac{2Gm(r)}{r}\right)^{-1}. \tag{5.37}$$

For $r > R$, $p(r) = 0$ and $m(r) = m$, and we can extend the right-hand side to $r > R$. We also require $\lim_{r\to\infty} A(r) = 1$ (see above). Integrating the above expression, we thus find

$$\log A(r) = -\int_r^\infty \left(8\pi Gp(s)s + \frac{2Gm(s)}{s^2}\right)\left(1 - \frac{2Gm(s)}{s}\right)^{-1}\mathrm{d}s, \tag{5.38}$$

from which Eq. (5.3) immediately follows. Note that $A(r)$ defined in this way is continuous at $r = R$ and reduces to $A(r)$ of the vacuum solution for $r > R$. This is because, in this case, Eq. (5.38) can be written as

$$\log A(r) = -\int_r^\infty \frac{2Gm}{s^2}\left(1-\frac{2Gm}{s}\right)^{-1} \mathrm{d}s = \log\left(1-\frac{2Gm}{r}\right). \tag{5.39}$$

□

5.2 Motion in Schwarzschild spacetime

5.2.1 Geodesics and general properties

The motion of a freely-falling test particle or photon in Schwarzschild spacetime is governed by the geodesic equations $\nabla_{\dot{c}}\dot{c} = 0$, where c denotes the worldline of the particle. Henceforth, we will set $G = 1$, as required in geometric units. In Exercise 2.10.8 it was shown that the geodesic equations in Schwarzschild spacetime can be written as:

$$\ddot{t} = -\frac{2m}{r^2 h(r)}\dot{t}\dot{r}, \tag{5.40}$$

$$\ddot{r} = -\frac{h(r)m}{r^2}\dot{t}^2 + \frac{m}{r^2 h(r)}\dot{r}^2 + rh(r)\dot{\theta}^2 + rh(r)\sin^2\theta\,\dot{\phi}^2, \tag{5.41}$$

$$\ddot{\theta} = \sin\theta\cos\theta\,\dot{\phi}^2 - \frac{2}{r}\dot{r}\dot{\theta}, \tag{5.42}$$

$$\ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi} - 2\cot\theta\,\dot{\theta}\dot{\phi}, \tag{5.43}$$

where $h(r) = 1 - 2m/r$ and the dot denotes differentiation with respect to the parameter λ that parametrizes the geodesic. If the geodesic is parametrized by proper time, we have the additional constraint $g(\dot{c},\dot{c}) = -1$ for particles (Sec. 3.2), which can be written as (cf. Exercise 2.10.8):

$$h(r)\dot{t}^2 - \frac{1}{h(r)}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 = 1. \tag{5.44}$$

For light (photons), we have $g(\dot{c},\dot{c}) = 0$ for any parameter λ parametrizing the geodesic, and we find the constraint

$$h(r)\dot{t}^2 - \frac{1}{h(r)}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 = 0. \tag{5.45}$$

Applied to astronomical objects, such test particles can be planets around a star or a star in the potential of a significantly more massive black hole. To a good approximation, the spacetime in our solar system is given by the exterior Schwarzschild solution; the application of the geodesic equations to our solar system led to many classic tests of general relativity, such as the deflection of light or the perihelion advance of Mercury, which we shall discuss in exercises. In this context, the following theorem proves to be very useful; it explores some general properties of the geodesic equations in Schwarzschild spacetime.

Theorem 5.2.1. *Let $c : I \subset \mathbb{R} \to M$ denote the timelike or null geodesic of a particle or photon in Schwarzschild spacetime. As above, $\dot{q}$ refers to differentiation of a quantity q with respect to the parameter λ that parametrizes the geodesic (proper time in the case of particles). Furthermore, let $h(r) = 1 - 2m/r$ and $u = 1/r$.*

(i) For particles and photons, there exist three constants of motion:

$$E \equiv h(r)\dot{t} = \text{const.}, \tag{5.46}$$
$$L \equiv r^2 \sin^2\theta\,\dot{\phi} = \text{const.}, \tag{5.47}$$

and

$$E^2 - 1 = \dot{r}^2 - \frac{2m}{r} + r(r-2m)\dot{\theta}^2 + L^2\frac{h(r)}{r^2\sin^2\theta} \quad \textit{(particles)}, \tag{5.48}$$
$$E^2 = \dot{r}^2 + r(r-2m)\dot{\theta}^2 + L^2\frac{h(r)}{r^2\sin^2\theta} \quad \textit{(photons)}. \tag{5.49}$$

(ii) Equatorial motion ($\theta = \pi/2$) of a particle satisfies the following differential equation:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{m}{L^2} + 3mu^2. \tag{5.50}$$

(iii) Equatorial motion ($\theta = \pi/2$) of a photon satisfies the following differential equations:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = 3mu^2, \tag{5.51}$$
$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{E^2}{L^2}r^4 - h(r)r^2. \tag{5.52}$$

Proof. (i)

$$\dot{E} = \frac{\mathrm{d}}{\mathrm{d}\lambda}(h(r)\dot{t}) = \frac{2m}{r^2}\dot{r}\dot{t} + h(r)\ddot{t} = 0, \tag{5.53}$$

where we have used Eq. (5.40) in the last step. Furthermore,

$$\dot{L} = \frac{\mathrm{d}}{\mathrm{d}\lambda}(r^2\sin^2\theta\dot{\phi}) \tag{5.54}$$
$$= r^2\sin^2\theta\ddot{\phi} + 2r\dot{r}\sin^2\theta\dot{\phi} + 2r^2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi} \tag{5.55}$$
$$= r^2\sin^2\theta\left(\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi}\right) \tag{5.56}$$
$$= 0, \tag{5.57}$$

where we have used Eq. (5.43) in the last step. Finally, with the definitions of E and L, one can write the constraints Eq. (5.44) and (5.45) as

$$h(r)^{-1}E^2 - h(r)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - \frac{L^2}{r^2\sin^2\theta} = \begin{cases} 1, & \text{particles} \\ 0, & \text{photons} \end{cases}. \tag{5.58}$$

Multiplying by $h(r)$ and rearranging terms, we obtain the desired identities.
(ii) Using the geodesic equation (Eq. (5.41))

$$\ddot{r} = -\frac{m}{r^2}\left(h(r)\dot{t}^2 - \frac{1}{h(r)}\dot{r}^2\right) + rh(r)\dot{\phi}^2 \tag{5.59}$$

and the constraint for particles (Eq. (5.44))

$$h(r)\dot{t}^2 - \frac{1}{h(r)}\dot{r}^2 = 1 + r^2\dot{\phi}^2 \tag{5.60}$$

we find:

$$\ddot{r} = -\frac{m}{r^2}\left(1 + r^2\dot{\phi}^2\right) + rh(r)\dot{\phi}^2 = -\frac{m}{r^2} + (r - 3m)\dot{\phi}^2. \tag{5.61}$$

Furthermore,

$$\ddot{u} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-\frac{\dot{r}}{r^2}\right) = -\frac{\ddot{r}r^2 - 2r\dot{r}^2}{r^4} = 2\frac{\dot{r}^2}{r^3} - \frac{\ddot{r}}{r^2}. \tag{5.62}$$

Equipped with this, and writing $\mathrm{d}u/\mathrm{d}\phi = \dot{u}/\dot{\phi}$, we conclude

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} = \frac{1}{\dot{\phi}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\dot{u}}{\dot{\phi}}\right) = \frac{\ddot{u}\dot{\phi} - \dot{u}\ddot{\phi}}{\dot{\phi}^3} \tag{5.63}$$

$$= 2\frac{\dot{r}^2}{r^3\dot{\phi}^2} - \frac{\ddot{r}}{r^2\dot{\phi}^2} + \frac{\dot{r}\ddot{\phi}}{r^2\dot{\phi}^3} \tag{5.64}$$

$$= \frac{\dot{r}^2}{r^3\dot{\phi}^2} + \frac{m}{r^4\dot{\phi}^2} - \frac{r - 3m}{r^2} - \frac{2\dot{r}^2}{r^3\dot{\phi}^2} \tag{5.65}$$

$$= \frac{m}{L} - u + 3mu^2. \tag{5.66}$$

(*iii*) The derivation of the first equation is left as an exercise (see below). From (*i*) we know that

$$\dot{r}^2 = E^2 - L^2h(r)/r^2. \tag{5.67}$$

Therefore, using the definition of L,

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{\dot{r}^2}{\dot{\phi}^2} = \frac{E^2}{L^2}r^4 - h(r)r^2. \tag{5.68}$$

□

Remark. In the proof of the previous theorem, we have chosen to derive the constants of motion directly from the geodesic equations in a 'pedestrian way'. There is an arguably more elegant way to derive them. Recall that the geodesic equations can be obtained from the Euler-Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0, \tag{5.69}$$

with the Lagrangian $\mathcal{L} = \frac{1}{2}g(\dot{c}, \dot{c}) = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ (cf. Exercise 2.10.6). The fact that $\mathcal{L}$ does not depend on t and ϕ in the case of Schwarzschild coordinates immediately yields E and L. The third constant of motion follows from the fact that $\mathcal{L}$ itself is a constant of motion (cf. Exercise 2.10.6).

Exercise 5.2.2. *Derive the first differential equation in Theorem 5.2.1 (iv).*

5.2.2 Periastron advance

In classical Newtonian physics, test bodies (such as planets) orbit their hosts on ellipses, obeying Kepler's laws. In general relativity, the periastron of such an orbit (perihelion in the case of our solar system) advances over time as we shall derive here. The perihelion advance was among the first classical tests of general relativity; in fact, Einstein obtained a precise prediction for the perihelion advance of Mercury even before he had finalized his field equations with matter (the perihelion result was presented to the Prussian Academy of Sciences on November 18, 1915)[4]. The significance of this result is that it provided an explanation for the mysterious advance of Mercury's orbit, which had already been found by Le Verrier in 1859 after having accounted for the effects of all other planets on Mercury. Einstein obtained an advance of $43''$ per century, which is in agreement with the observational value of $(45\pm5)''$ per century known to astronomers at the time.[5]

In order to see how the periastron advance arises, let us assume that the gravitational field of the host star can be described by the exterior Schwarzschild solution (Sec. 5.1). This will be further justified in Sec. 7.2. The orbital motion of a test body such as Mercury is then governed by the differential equation (see Theorem 5.2.1):

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{m}{L^2} + 3mu^2, \tag{5.70}$$

where $u = 1/r$. In Newtonian gravity, the same differential equation holds, but without the second term on the right-hand side. Here, we shall assume that this term is small, i.e., that there are only small deviations from the Newtonian case. This is true, in particular, for the solar system, since

$$\frac{3mu^2}{u} = \frac{3R_\mathrm{S}}{2r} \lesssim \frac{3}{2}\frac{R_\mathrm{S}}{R_\odot} \sim 10^{-6}. \tag{5.71}$$

We shall formulate the derivation of the periastron advance as the following

Exercise 5.2.3. ***(Periastron advance)***

(a) Show that the ***Kepler ellipse***

$$u = \frac{m}{L^2}(1 + \epsilon\cos\phi) \tag{5.72}$$

with eccentricity ϵ is a solution of the differential equation (5.70) in the Newtonian case. Note that the periastron of the ellipse is a maximum of $u(\phi)$; it is thus located at $\phi = 0$ in this case.

(b) Inserting the Newtonian solution from (a) into the second term on the right-hand side of Eq. (5.70) leads to an inhomogeneous differential equation of second order, with

$$u(\phi) = \frac{m}{L^2}(1 + \epsilon\cos\phi) + \frac{3m^3}{L^4}\left(1 + \frac{\epsilon^2}{2} - \frac{\epsilon^2}{6}\cos 2\phi + \epsilon\phi\sin\phi\right) \tag{5.73}$$

[4] A. Einstein, "Explanation of the Perihelion Motion of Mercury from the General Theory of Relativity", Proceedings of the Royal Prussian Academy of Sciences (1915), 831-839. See also: The Collected Papers of Albert Einstein, Vol. 6, Doc. 24, https://einsteinpapers.press.princeton.edu/papers

[5] See last page of A. Einstein, "Explanation of the Perihelion Motion of Mercury from the General Theory of Relativity", Proceedings of the Royal Prussian Academy of Sciences (1915), 831-839. See also: The Collected Papers of Albert Einstein, Vol. 6, Doc. 24, https://einsteinpapers.press.princeton.edu/papers

as a particular solution. This solution represents an approximate solution to Eq. (5.70), assuming that the second term on the right-hand side only represents a small deviation from the Newtonian case. Show that the periastron advances from $\phi = 0$ to $\phi = 2\pi + \delta$, where the periastron advance is given by

$$\delta = \frac{6\pi m^2}{L^2}. \tag{5.74}$$

Re-expressing in terms of the semi-major axis a and eccentricity ϵ of an ellipse via the relation

$$a(1-\epsilon^2) = \frac{L^2}{m}, \tag{5.75}$$

one can rewrite Eq. (5.74) as

$$\delta = 3\pi \frac{R_\mathrm{S}}{a(1-\epsilon^2)}, \tag{5.76}$$

where R_S is the Schwarzschild radius (Eq. (5.4)). This is the famous formula obtained by Einstein in November 1915[6]. We note, however, that Einstein derived this result in a different way (using a post-Newtonian approximation based on the (correct) vacuum equations), as the final field equations and the Schwarzschild solution had not yet been discovered. As Einstein already pointed out in his paper, the periastron advance is most pronounced for large eccentricities and/or small semi-major axes. Mercury is thus most susceptible to this general relativistic effect in our solar system. Finally, it is important to realize that the periastron advance probes the non-linearities of general relativity; it vanishes in the linearized theory (Chapter 7). This is in contrast to some other classic tests of general relativity, such as the deflection of light, which can be obtained already on the linearized level.

5.2.3 Deflection of light

The fact that photons travel on null geodesics in general relativity (i.e., $\nabla_{\dot{c}}\dot{c} = g(\dot{c},\dot{c}) = 0$) leads to the phenomenon of deflection of light in the presence of a gravitational field, which we shall discuss here. We shall focus on the deflection of light in Schwarzschild spacetime—the basis for some classic tests of general relativity and gravitational lensing.

A light ray in Schwarzschild spacetime is described by (cf. Theorem 5.2.1)

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = 3mu^2, \tag{5.77}$$

where $u = 1/r$. In the Newtonian limit, the right-hand side of this equation is zero and one recovers the corresponding homogeneous differential equation. Let us consider a photon that passes by an astronomical object of mass m with impact parameter r_0. Let ϕ_∞ denote the half-deflection angle, i.e., half the angle between the original direction of the photon and its asymptotic direction after having passed the astronomical object. As in the case of the periastron advance, we assume that the term on the right-hand side of Eq. (5.77) is small, so that it only represents a perturbation of the Newtonian result. This approximation is valid for the solar system as demonstrated by Eq. (5.71). We calculate the deflection angle in the following

[6] cf. Eq. (13) in A. Einstein "Explanation of the Perihelion Motion of Mercury from the General Theory of Relativity", Proceedings of the Royal Prussian Academy of Sciences (1915), 831-839. See also: The Collected Papers of Albert Einstein, Vol. 6, Doc. 24, https://einsteinpapers.press.princeton.edu/papers

Exercise 5.2.4. ***(Deflection of light)***

(a) *Show that the straight line $u = r_0^{-1}\sin\phi$ is a solution to the corresponding homogeneous differential equation of Eq. (5.77).*

(b) *Inserting the solution of (a) into the right-hand side of Eq. (5.77), one obtains an inhomogeneous differential equation with particular solution*

$$u_{\text{part.}}(\phi) = \frac{3m}{2r_0^2}\left(1 + \frac{1}{3}\cos 2\phi\right). \tag{5.78}$$

Composing an approximate solution to Eq. (5.77) by adding this particular solution to the solution of the homogeneous equation, derive the following approximate expression for ϕ_∞, which is reached as the particle goes to $r \to \infty$:

$$\phi_\infty = -\frac{2m}{r_0}. \tag{5.79}$$

The **total deflection angle** is then given by

$$\delta = 2|\phi_\infty| = \frac{4m}{r_0} = \frac{2R_\text{S}}{r_0} = 1.75'' \frac{R_\odot}{r_0}. \tag{5.80}$$

It can be shown that, in contrast to the periastron advance (Sec. 5.2.2), the correct result can already be obtained within the linearized theory (Sec. 7.2); therefore, tests of general relativity based on the deflection of light do not probe the non-linearities of the theory.

Einstein predicted the value of 1.7" for the Sun already in November 1915, just before he published the final field equations.[7] This represents a correction of his previous result from 1911, which differs by a factor of 1/2.[8] The reason for this difference is that the earlier result is a purely Newtonian result that does not include effects of spatial curvature (see also Sec. 7.2).

Deflection of light in the presence of a gravitational field manifests itself in shifting the actual positions of stars if their light rays pass close to another astronomical object on their way to the observer. This effect can be exploited during a total solar eclipse: stars close to the solar limb are then visible and their positions appear to be radially shifted outward as compared to their normal positions on the night sky. Einstein already suggested to conduct such observations during a solar eclipse in his 1911 paper. The first successful observations were made during the total solar eclipse on May 29, 1919. The 1919 solar eclipse campaign comprised two teams: Sir Arthur Eddington led one team on the island of Principe off the coast of present-day Equatorial Guinea and the other team was led by Andrew Crommelin in the city of Sobral in northern Brazil. The results obtained were

$$\delta = \begin{cases} (1.6 \pm 0.3)'', & \text{Principe} \\ (1.98 \pm 0.12)'', & \text{Sobral} \end{cases}, \tag{5.81}$$

in agreement with Einstein's prediction.

[7] See below Eq. (5) in A. Einstein, "Explanation of the Perihelion Motion of Mercury from the General Theory of Relativity", Proceedings of the Royal Prussian Academy of Sciences (1915), 831-839. See also: The Collected Papers of Albert Einstein, Vol. 6, Doc. 24, https://einsteinpapers.press.princeton.edu/papers

[8] A. Einstein, "On the Influence of Gravitation on the Propagation of Light", Annalen der Physik 35 (1911), 898-908. See also: The Collected Papers of Albert Einstein, Vol. 3, Doc. 23, https://einsteinpapers.press.princeton.edu/papers

This campaign represents the first observational verification of general relativity by an independent team of scientists, it popularized Einstein's theory, and made him famous essentially over night. The New York Times headline on November 7, 1919 was "Revolution in Science/ New Theory of the Universe/ Newtonian Ideas Overthrown." A detailed discussion of the 1919 campaign can be found in Will (2015).

5.3 Non-rotating black holes

In this section, we shall analyze the Schwarzschild vacuum solution (Eq. (5.1)) for a point-like mass and interpret it as the spacetime of a non-rotating black hole. In the following, we will set $G = 1$, as required in geometric units.

We shall base the discussion on the radial motion of a particle or photon in Schwarzschild spacetime. Due to spherical symmetry the essence of radial motion is captured by the r-t subspace (plane) of M,

$$P = \mathbb{R} \times ((0, 2m) \cup (2m, \infty)), \tag{5.82}$$

with restricted metric

$$(g_{\mu\nu}) = \mathrm{diag}\left(-\left(1 - \frac{2Gm}{r}\right), \left(1 - \frac{2Gm}{r}\right)^{-1}\right) \tag{5.83}$$

and line element

$$\mathrm{d}s^2 = -\left(1 - \frac{2Gm}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2Gm}{r}\right)^{-1}\mathrm{d}r^2. \tag{5.84}$$

The geodesic equations (5.40)–(5.43) reduce to ($\dot\theta = \dot\phi = 0$)

$$\ddot t = \frac{2m}{r(2m - r)}\dot t \dot r, \tag{5.85}$$

$$\ddot r = \frac{(2m - r)m}{r^3}\dot t^2 + \frac{m}{r(r - 2m)}\dot r^2, \tag{5.86}$$

with constraints (cf. Eq. (5.44))

$$(r - 2m)\dot t^2 + \frac{r^2}{2m - r}\dot r^2 = r \tag{5.87}$$

for particles and (cf. Eq. (5.45))

$$(r - 2m)^2 \dot t^2 = r^2 \dot r^2 \tag{5.88}$$

for photons.

Light cones. Let us analyze the vacuum Schwarzschild solution across $r = R_\mathrm{S}$. Let $v = \lambda_t \partial_t + \lambda_r \partial_r$ be a tangent vector at some point p along the geodesic c. This vector is time-like if and only if

$$g(v, v) < 0 \quad \Leftrightarrow -(1 - 2m/r)\lambda_t^2 + (1 - 2m/r)^{-1}\lambda_r^2 < 0, \tag{5.89}$$

or, equivalently,

$$g(v, v) < 0 \quad \Leftrightarrow \begin{cases} \left|\frac{\lambda_r}{\lambda_t}\right| < (1 - 2m/r), & r > 2m \\ \left|\frac{\lambda_r}{\lambda_t}\right| > (2m/r - 1), & r < 2m \end{cases}. \tag{5.90}$$

From Eq. (5.84), we also find for the slope of the light cone (for null directions, $\mathrm{d}s = 0$):

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(1 - \frac{2m}{r}\right)^{-1} . \tag{5.91}$$

For large $r \to \infty$, the slope is $\mathrm{d}t/\mathrm{d}r = \pm 1$, as in Minkowski space. As the Schwarzschild solution approaches flat space at large radii, it is reasonable to choose the time orientation of the Schwarzschild spacetime accordingly, i.e., that timelike vectors v with $\lambda_t > 0$ are future oriented. As one approaches $r = 2m$ from large radii, the slope (5.91) diverges, $\mathrm{d}t/\mathrm{d}r \to \pm\infty$, and the light cones become increasingly narrow. For $r < 2m$, $|\lambda_r/\lambda_t| > 1$ for timelike vectors, i.e., vectors with $\lambda_r = 0$ cannot be timelike anymore, and the sign of λ_r determines whether timelike vectors point to the future or the past. We set (for reasons that will become obvious in Sec. 5.5) timelike vectors with $\lambda_r < 0$ to be future oriented. This means that future-oriented light cones of particles and photons point toward the singularity at $r = 0$, i.e., once they reached the region $r < 2m$, they will not be able to escape that region anymore. Note also that the slope of the light cone diverges when approaching the singularity, $\mathrm{d}r/\mathrm{d}t \to \pm\infty$, i.e., that the light cones become again increasingly narrow.

Theorem 5.3.1. *Light rays in the Schwarzschild r-t plane follow the trajectories*

$$t(r) = \pm r^*(r) + \text{const.}, \tag{5.92}$$

where

$$r^*(r) = r + 2m \log|r - 2m| \tag{5.93}$$

Proof. The slope for null directions (Eq. (5.91)), written as

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\frac{r}{r - 2m} = \pm\left(1 + \frac{2m}{r - 2m}\right), \tag{5.94}$$

yields after integration:

$$t(r) = \pm(r + 2m \log|r - 2m|) + \text{const.}. \tag{5.95}$$

□

We note that the plus and minus sign in Eq. (5.92) correspond to outgoing and ingoing light rays, respectively (coordinate time is increasing/decreasing with increasing radius).

Discussion. Let us discuss the behavior of the above light ray solution for $r \to \pm\infty$ and $r > 2m$. Clearly, $t \to \pm\infty$, which means that an ingoing light ray does not reach the Schwarzschild horizon at $r_\mathrm{S} = 2m$ in a finite time as seen by a distant observer at rest (for that observer, proper time is given by the coordinate time t). Likewise, outgoing photons emitted near the horizon will not reach the observer in a finite amount of coordinate time. The situation is even worse for particles, for which $\dot{c}$ is within the light cone, so $\mathrm{d}t/\mathrm{d}r$ diverges even faster. Consider a particle on a radial geodesic approaching $r = 2m$; assume that it sends light signals to a distant observer at $r = r_0$. For any $r_0 > 2m$, t_a, where t_a is the arrival time of the photon at the observer, diverges fast to infinity. This leads a distant observer to conclude that the particle never reaches the horizon. It should also be noted that due to the gravitational redshift, these light signals also get weaker as the particle approaches the horizon. However, an observer co-moving with the particle will experience a different scenario, as we shall now discuss. The discrepancy between these conclusions is due to the fact that the Schwarzschild coordinates exhibit a coordinate singularity at $r = 2m$, i.e., they are not suited to parametrize the spacetime around $r = 2m$. This will be discussed further in Sec. 5.5.

Radial infall. We shall now show that a particle at rest at $r = \infty$ crosses the Schwarzschild horizon and falls into the singularity in a finite amount of time. This is summarized in the following

Theorem 5.3.2. *Let $\tau(r)$ denote the proper time that elapses while a particle falls radially from $r_0 > r > 0$ to r in Schwarzschild spacetime with zero initial radial velocity. Then its radius and proper time can be parametrized by*

$$r(\eta) = \frac{r_0}{2}(1 + \cos\eta), \tag{5.96}$$

$$\tau(\eta) = \frac{r_0}{2}\sqrt{\frac{r_0}{2m}}(\eta + \sin\eta), \tag{5.97}$$

with $0 < \eta < \pi$.

Proof. With the assumption $\dot{r} = 0$, we conclude from Theorem 5.2.1 that

$$E^2 - 1 = -\frac{2m}{r_0} = \text{const.} \tag{5.98}$$

Additionally, with

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}r/\mathrm{d}\eta}{\mathrm{d}\tau/\mathrm{d}\eta} = -\frac{\sin\eta}{\sqrt{r_0/2m}(1 + \cos\eta)} \tag{5.99}$$

one can write the right-hand side of Eq. (5.48) as

$$\dot{r}^2 - \frac{2m}{r} = \frac{2m}{r_0}\frac{\sin^2\eta}{(1 + \cos\eta)^2} - \frac{4m}{r_0(1 + \cos\eta)} \tag{5.100}$$

$$= \frac{2m}{r_0}\frac{\sin^2\eta - 2(1 + \cos\eta)}{1 + 2\cos\eta + \cos^2\eta} = -\frac{2m}{r_0}. \tag{5.101}$$

□

We note that according to the parametrization of radial infall as discussed in this theorem, nothing unusual occurs at $r = 2m$. The particle starts at $r(\eta = 0) = r_0$ and reaches the central singularity at $r = 0$ at the finite proper time

$$\tau(\eta = \pi) = \frac{\pi}{2}r_0\sqrt{\frac{r_0}{2m}}. \tag{5.102}$$

5.4 Apparent size of astronomical objects

Another consequence of geodesic motion in general relativity is that compact objects such as neutron stars and black holes have an apparent size that differs from their actual size. This is due to the deflection of light in the vicinity of these objects (see also Sec. 5.2.3). Such distortion effects are important, e.g., in the context of 'black-hole imaging' with the Event Horizon Telescope (EHT; see Event Horizon Telescope Collaboration et al. 2019a,b,c,d,e,f for first results) or when analyzing soft X-ray emission from neutron stars to measure their masses and radii (thus constraining the equation of state of nuclear matter at high densities; see, e.g., Riley et al. 2019; Raaijmakers et al. 2019; Miller et al. 2019 for first results from the NICER collaboration in this regard). We shall focus solely on calculating the apparent size of compact objects as seen by a distant observer in this section. The discussion here neglects additional complexity that arises in full 3D imaging combined with related effects such as a simultaneous gravitational redshift, which need to be tackled by numerical ray-tracing techniques in realistic astrophysical situations. However, the simplified discussion here illustrates some of the basic concepts and may provide a flavor or whet the appetite for more complex 'compact-object optics' in realistic astrophysical scenarios.

It is trivial knowledge that the angular size of a star in Euclidean geometry is given by

$$\sin\beta = \frac{R}{r_0}, \tag{5.103}$$

where R is the radius of the star and r_0 the distance to the observer. We shall now investigate how this result changes in general relativity due to deflection of light. We start by computing the apparent size of stars ($R > 3m$) and then proceed to black holes ($R < 2m$). We shall assume that spacetime around compact objects is described by the exterior Schwarzschild spacetime (Sec. 5.1). This assumption is valid as long as significant rotation of the object can be neglected. Due to spherical symmetry, one can always rotate the coordinate frame such that a distant observer is in the equatorial plane of the Schwarzschild spacetime. From Theorem 5.2.1 we then know that the motion of photons in that plane follows the differential equation

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{E^2}{L^2}r^4 - h(r)r^2. \tag{5.104}$$

Since the left-hand side must be greater or equal zero, the photon radius is confined to

$$f(r) \equiv \frac{\sqrt{h(r)}}{r} \leq \frac{E}{|L|}, \tag{5.105}$$

where, as usual, $h(r) = 1 - 2m/r$. We restrict the discussion to outside the Schwarzschild radius $r \geq R_\mathrm{S} = 2m$. The function $f(r)$ vanishes at $r = 2m$, reaches a maximum of $f(r_\mathrm{max}) = (3\sqrt{3}m)^{-1}$ at $r_\mathrm{max} = 3m$ and then asymptotes to zero for $r \to \infty$. We base the following discussion on a useful property:

Theorem 5.4.1. *Let $\dot{c} = \dot{t}\partial_t + \dot{r}\partial_r + \dot{\phi}\partial_\phi$ be the tangent vector (4-velocity) of a photon in the Schwarzschild equatorial plane ($\theta = \pi/2$). For $r > R_\mathrm{S} = 2m$ the angle α between $\dot{r}\partial_r + \dot{\phi}\partial_\phi$ and $-\partial_r$ in the Euclidean sense is given by*

$$\sin\alpha = \frac{|L|}{E}f(r). \tag{5.106}$$

We note that $\alpha = \pi - \alpha_r$, where α_r is the angle between the spatial part of the tangent vector ($\dot{r}\partial_r + \dot{\phi}\partial_\phi$) and the radial direction.

Proof.

$$\begin{aligned}\sin\alpha &= \frac{|\dot\phi\partial_\phi|}{|\dot r\partial_r+\dot\phi\partial_\phi|}=\left(\frac{g(\dot\phi\partial_\phi,\dot\phi\partial_\phi)}{g(\dot r\partial_r+\dot\phi\partial_\phi,\dot r\partial_r+\dot\phi\partial_\phi)}\right)^{1/2} && (5.107)\\ &= \frac{|\dot\phi|\sqrt{g_{\phi\phi}}}{\sqrt{\dot r^2 g_{rr}+\dot\phi^2 g_{\phi\phi}}}=\frac{|\dot\phi|r}{\sqrt{\dot r^2 h(r)^{-1}+\dot\phi^2r^2}} && (5.108)\\ &= \frac{|\dot\phi|r}{\sqrt{h(r)}\dot t}=\frac{|L|/r}{E/\sqrt{h(r)}}, && (5.109)\end{aligned}$$

where we have used $\sin\theta=1$. We have inserted the constraint for photons (Eq. (5.45)) into Eq. (5.108) and used the definitions of E and L (cf. Theorem 5.2.1). □

We can now compute the angular size of a star in general relativity:

Theorem 5.4.2. *Consider a star of gravitational mass m, radius R, with $3m<R<r_0$, where r_0 is the distance to the observer. Then its apparent angular size is given by*

$$\sin\beta=\frac{R}{r_0}\sqrt{\frac{h(r_0)}{h(R)}}. \tag{5.110}$$

Proof. The idea is to calculate the constant $|L|/E$ for a tangential photon, a light ray that connects the observer to the limb of the star, and then use Theorem 5.4.1. For such a tangential photon, $\mathrm{d}r/\mathrm{d}\phi=0$, and from Eq. (5.105) one finds $|L|/E=R/\sqrt{h(R)}$. Evaluating the formula of Theorem 5.4.1 at the observer location $r=r_0$ then yields the desired result. One can further convince oneself that $|L|/E=R/\sqrt{h(R)}$ indeed corresponds to such a tangential photon. If $|L|/E>R/\sqrt{h(R)}$, then the constraint Eq. (5.105) restricts the photon to a radius $r>r_{\min}$, where $r_{\min}>R$ is given by $f(r_{\min})=E/|L|$. Therefore, the photon cannot originate from the star. In contrast, if $|L|/E<R/\sqrt{h(R)}$, then the light ray extends down to $r_{\min}<R$ according to Eq. (5.105) and the photon originates from $r<R$. □

For normal stars, the correction factor $\sqrt{h(r_0)/h(R)}$ in Eq. (5.110) with respect to the Euclidean case (Eq. (5.103)) is minute. However, for compact stars such as neutron stars, the correction can be significant ($\sim$10%). We now turn to black holes.

Theorem 5.4.3. ***(Photon sphere)***
*Consider a non-rotating black hole of mass m as described by the vacuum Schwarzschild solution. There exists a **photon sphere** at $r=3m$, i.e., a sphere with closed photon orbits (**photon rings**). These orbits satisfy $L/E=3\sqrt{3}m$.*

Proof. Due to spherical symmetry, it is sufficient to show the existence of photon rings. That is, we can reduce the problem to a photon trajectory with $\theta=\text{const.}=\pi/2$ and show that there exist closed orbits at $r=3m$. In order to find such photon rings, we start with the following ansatz for the photon trajectory c:

$$c(\lambda)=(t(\lambda),r_0,\pi/2,\lambda), \tag{5.111}$$

where λ parametrizes, as usual, the null geodesic and r_0 is a fixed radius. From the photon constraint Eq. (5.45) with $\dot{r} = \dot{\theta} = 0$, $\dot{\phi} = 1$, one finds $\dot{t} = r_0/\sqrt{h(r_0)}$. Inserting this result into the geodesic equation for r (Eq. (5.41)) and using that $\ddot{r} = 0$, we find $r_0 = 3m$. This shows the existence of closed photon orbits. Furthermore, for such orbits:

$$\frac{L}{E} = \frac{r_0^2\dot{\phi}}{h(r_0)\dot{t}} = \frac{(3m)^2}{3m\sqrt{h(3m)}} = 3\sqrt{3}m. \tag{5.112}$$

□

Exercise 5.4.4. *Provide an alternative proof to Theorem 5.4.3. Start by considering a photon at $r = r_0 = 3m$ that starts with an angle α smaller and larger than $\pi/2$ relative to $-\partial_r$ (cf. Theorem 5.4.1) and infer its fate by using Theorem 5.4.1 and the properties of $f(r)$.*

Theorem 5.4.5. *Consider a non-rotating black hole with mass m as described by the vacuum Schwarzschild solution. The angular size of the black hole for an observer at $r_0 > R_S = 2m$ is given by*

$$\sin\beta = \frac{3m\sqrt{3h(r_0)}}{r_0}, \tag{5.113}$$

with $\beta < \pi/2$ for $r_0 > 3m$ and $\beta > \pi/2$ for $r_0 < 3m$.

Proof. We shall only consider the case $r_0 > 3m$—the argument is analogous for $2m < r_0 < 3m$. Let us assume that we shoot a photon from r_0 toward the black hole with $L/E < 3m\sqrt{3}$. Then according to Theorem 5.4.1,

$$\sin\alpha < 3m\sqrt{3}\frac{1}{3m\sqrt{3}} = 1, \tag{5.114}$$

recalling that $(3m\sqrt{3})^{-1}$ is the maximum of $f(r)$ as mentioned above. Therefore, the angle with respect to $-\partial_r$ is bounded by $\alpha < \pi/2$ along the geodesic (note that L/E is a constant of motion). Theorem 5.4.3 and Exercise 5.4.4 then show that upon reaching $r = 3m$ the photon must fall into the black hole.

In contrast, if the photon is shot toward the black hole with $L/E > 3m\sqrt{3}$, then Eq. (5.105) implies that $f(r) < (3m\sqrt{3})^{-1}$, i.e., that $r > 3m$. Hence, the photon will not fall into the black hole.

Thus the limiting case of a tangential photon is given by $L/E = 3m\sqrt{3}$. Inserting this into the formula of Theorem 5.4.1 yields the desired result. □

Comparing the formula in Theorem 5.4.5 with Eq. (5.103), one concludes that for a distant observer ($h(r_0) \to 1$) the apparent radius of the black hole is

$$R_{BH} = 3m\sqrt{3}. \tag{5.115}$$

Intrinsically, the black hole appears as a black (i.e., totally absorbing) object of radius $3m$, owing to the photon sphere. According to Eq. (5.115), deflection of light increases the radius by a factor of $\sqrt{3}$ for distant observers, i.e., by ∼73%.

5.5 Kruskal continuation of Schwarzschild spacetime

The metric tensor of the Schwarzschild solution is singular at the Schwarzschild radius $r = R_S = 2m$ (cf. Sec. 5.1). In Sec. 5.3, it was shown that a particle or photon does not reach this singularity in finite Schwarzschild coordinate time. However, it was also shown that such a particle or photon, in fact, reaches this singularity and the central singularity at $r = 0$ in finite proper time. Furthermore, the analysis of radial infall as seen by a comoving observer following the cycloid showed that nothing unusual happens at $r = 2m$. Indeed, one can compute curvature invariants such as the Kretschmann scalar (see exercise below) and conclude that they remain finite at $r = 2m$, i.e., that, again, there is nothing unusual about this coordinate sphere (except for aspects of causality, as discussed in Sec. 5.3). The curvature scalar R, another curvature invariant, is not suitable for this discussion as it vanishes in Schwarzschild spacetime (the vacuum Schwarzschild solution is Ricci-flat by construction, see Sec. 5.1).

Exercise 5.5.1. *Show that the* ***Kretschmann scalar*** *for Schwarzschild spacetime is given by*

$$K \equiv R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} = 12 \frac{R_S^2}{r^6}, \tag{5.116}$$

where $R_S = 2m$ is the Schwarzschild radius and $R_{\mu\nu\rho\sigma}$ the covariant curvature tensor. Why is the singularity of K for $r \to 0$ a true spacetime singularity?

The discrepancy between what a distant Schwarzschild observer and a comoving observer experience when describing the radial infall of a photon or particle suggests that the singularity of the Schwarzschild solution at $r = 2m$ is purely a coordinate singularity, i.e., the singularity only exists because the chosen coordinate system is not applicable there. Indeed, one can easily construct coordinate singularities in spaces that have no pathological features, as the following exercise demonstrates.

Exercise 5.5.2. *Consider the manifold $M = \mathbb{R}$ with Cartesian coordinates. The metric is written as $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. Consider the coordinate transformation $x \to x' \equiv x^3$, $y \to y' \equiv y$ and show that this generates a singularity at $x = y = 0$.*

We shall now construct a chart that is continuous across $r = 2m$ and that covers the entire Schwarzschild spacetime interior and exterior to $r = 2m$. Let us first focus on the exterior region $r > 2m$.

Exterior Schwarzschild region ($r > 2m$):

(1) Consider the coordinate transformation $(t, r) \to (t, r^*)$, where

$$r^* = r + 2m \log(|r - 2m|). \tag{5.117}$$

This transformation extends the r-t half plane $r > 2m$ to the entire r^*-t-plane. From Theorem 5.3.1, we conclude that the outgoing null geodesics (photon trajectories) correspond to the straight lines $t = r^* + \text{const.}$, whereas the orthogonal lines $t + r^* = \text{const.}$ correspond to the ingoing null geodesics.

(2) Consider the coordinate transformation $(t, r^*) \to (U, V)$, where

$$U = t - r^* \tag{5.118}$$

$$V = t + r^*. \tag{5.119}$$

This is a rotation-reflection modulo a prefactor $\sqrt{2}$. The outgoing null geodesics are $U=$ const. and the ingoing null geodesics are $V=$ const..

(3) Consider the coordinate transformation $(U,V)\to(\tilde{u},\tilde{v})$, where

$$\tilde{u} = -e^{-U/4m} \tag{5.120}$$
$$\tilde{v} = e^{V/4m}. \tag{5.121}$$

This is a bijective mapping of the entire U-V plane to second quadrant of the $\tilde{u}$-$\tilde{v}$ plane. The outgoing and ingoing null geodesics are still straight lines parallel to the major coordinate axes, but are now restricted to half lines ($\tilde{u}<0$, $\tilde{v}>0$).

(4) Consider the final coordinate transformation $(\tilde{u},\tilde{v})\to(u,v)$, where

$$u = \frac{1}{2}(\tilde{v}-\tilde{u}) \tag{5.122}$$
$$v = \frac{1}{2}(\tilde{v}+\tilde{u}). \tag{5.123}$$

Essentially, this is again a rotation-reflection of the second quadrant in the $\tilde{u}$-$\tilde{v}$ plane into the cone $|u|>|v|$, $u>0$ with 90° opening angle in the u-v plane (Region I in Fig. **??**). The outgoing and ingoing null geodesics are straight lines with ±1 slope ($\pm45°$): $v=u-$const. for outgoing and $u+v=$ const. for ingoing photons.

Altogether, we find the concatenated coordinate transformation:

$$u = \frac{1}{2}\left(e^{\frac{t+r^*}{4m}}+e^{\frac{-t+r^*}{4m}}\right) = \cosh\frac{t}{4m}e^{\frac{r}{4m}}e^{\frac{1}{2}\log(r-2m)} = \sqrt{r-2m}\,e^{\frac{r}{4m}}\cosh\frac{t}{4m} \tag{5.124}$$
$$v = \frac{1}{2}\left(e^{\frac{t+r^*}{4m}}-e^{\frac{-t+r^*}{4m}}\right) = \sinh\frac{t}{4m}e^{\frac{r}{4m}}e^{\frac{1}{2}\log(r-2m)} = \sqrt{r-2m}\,e^{\frac{r}{4m}}\sinh\frac{t}{4m}. \tag{5.125}$$

These expressions immediately imply:

$$u^2-v^2 = (r-2m)e^{r/2m}, \tag{5.126}$$
$$\frac{v}{u} = \tanh\left(\frac{t}{4m}\right). \tag{5.127}$$

Hence, lines of $r=$ const. correspond to hyperbolas in the u-v plane (branch with $u>0$), which approach the diagonal at 45° for $r\to2m$, while lines of $t=$ const. correspond to radial half lines with slope $\tanh(t/4m)$.

Metric components. Let us now compute the metric components in the Kruskal coordinates (u,v). We start by computing the partial coordinate derivatives:

$$\frac{\partial u}{\partial t}=\frac{v}{4m},\quad \frac{\partial v}{\partial t}=\frac{u}{4m},\quad \frac{\partial u}{\partial r}=\frac{u}{4m}+\frac{u}{2r-4m}=\frac{u}{4mh(r)},\quad \frac{\partial v}{\partial r}=\frac{v}{4mh(r)}. \tag{5.128}$$

Denoting $x^\mu=(t,r)$ and $\bar{x}^\mu=(u,v)$, Eq. (2.34) relates the corresponding coordinate basis vector fields:

$$\partial_t = \frac{\partial\bar{x}^j}{\partial t}\bar{\partial}_j = \frac{v}{4m}\partial_u+\frac{u}{4m}\partial_v, \tag{5.129}$$
$$\partial_r = \frac{\partial\bar{x}^j}{\partial r}\bar{\partial}_j = \frac{u}{4mh(r)}\partial_u+\frac{v}{4mh(r)}\partial_v. \tag{5.130}$$

Solving for ∂_u and ∂_v, one finds:

$$\partial_u = -\frac{4m}{re^{r/2m}}\left(\frac{v}{h(r)}\partial_t - u\partial_r\right), \quad (5.131)$$

$$\partial_v = \frac{4m}{re^{r/2m}}\left(\frac{u}{h(r)}\partial_t - v\partial_r\right). \quad (5.132)$$

Equipped with this, one obtains the metric components:

$$g_{uu} = g(\partial_u, \partial_u) = \frac{16m^2}{r^2 e^{r/m}}\left(\frac{v^2}{h(r)^2}g_{tt} + u^2 g_{rr}\right) = \frac{16m^2}{r^2 e^{r/m}}\frac{u^2 - v^2}{h(r)} = \frac{16m^2}{re^{r/2m}} \quad (5.133)$$

$$g_{vv} = g(\partial_v, \partial_v) = \frac{16m^2}{r^2 e^{r/m}}\left(\frac{u^2}{h(r)^2}g_{tt} + v^2 g_{rr}\right) = \frac{16m^2}{r^2 e^{r/m}}\frac{v^2 - u^2}{h(r)} = -\frac{16m^2}{re^{r/2m}}, \quad (5.134)$$

$$g_{uv} = 0. \quad (5.135)$$

where we have also made use of Eq. (5.126). In summary, we can thus write the line element in Kruskal coordinates as

$$\mathrm{d}s^2 = -f^2(u,v)(\mathrm{d}v^2 - \mathrm{d}u^2) + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \quad (5.136)$$

where r is implicitly defined as a function of u and v (cf. Eqs. (5.126)–(5.127)) and

$$f^2(u,v) = \frac{16m^2}{re^{r/2m}} > 0. \quad (5.137)$$

Note that the two-dimensional (u,v) part of the metric in Kruskal coordinates $\bar{g}^{\theta,\phi}_{\mu\nu}$ (the submanifolds for $\theta, \phi = \text{const.}$) is conformally equivalent to the Minkowski metric $\eta_{\mu\nu}$ with line element $\mathrm{d}s^2 = -\mathrm{d}u^2 + \mathrm{d}v^2$, i.e., $\bar{g}^{\theta,\phi}_{\mu\nu}$ can be obtained via a **conformal transformation**

$$\eta_{\mu\nu} \to \bar{g}^{\theta,\phi}_{\mu\nu} = \omega^2(u,v)\eta_{\mu\nu} \quad (5.138)$$

with conformal factor $\omega^2(u,v) = -f^2(u,v)$. For radially emitted light rays ($\mathrm{d}s^2 = \mathrm{d}\theta = \mathrm{d}\phi = 0$), one has

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \pm 1, \qquad \text{if } f^2(u,v) \neq 0. \quad (5.139)$$

We thus conclude (as already noted above) that light rays correspond to lines at $\pm 45°$ as in Minkowski spacetime. Timelike tangent vectors $x = \lambda_u \partial_u + \lambda_v \partial_v$ must satisfy

$$g(x,x) = \lambda_u^2 f^2(u,v) + \lambda_v^2 f^2(u,v) \overset{!}{<} 0 \quad \Leftrightarrow \quad |\lambda_v| > |\lambda_u|. \quad (5.140)$$

Following the steps (1)–(4) of the construction of Kruskal coordinates, one finds that ∂_v is timelike and future oriented. Therefore, all timelike vectors x with $b > 0$ are future oriented. Let us now extend the discussion to the interior Schwarzschild region.

Interior Schwarzschild region ($r < 2m$): In this region of the Schwarzschild solution, one proceeds as for $r > 2m$ with the construction of Kruskal coordinates, with the exception of Step (3), where the transformations differ slightly by a minus sign:

(3)′ Consider the coordinate transformation $(U, V) \to (\tilde{u}, \tilde{v})$, where

$$\tilde{u} = e^{-U/4m} \quad (5.141)$$
$$\tilde{v} = e^{V/4m}. \quad (5.142)$$

Altogether the series of transformations leads to the following expressions for (u, v):

$$u = \sqrt{2m - r}\, e^{\frac{r}{4m}} \sinh \frac{t}{4m}, \qquad (r < 2m), \quad (5.143)$$
$$v = \sqrt{2m - r}\, e^{\frac{r}{4m}} \cosh \frac{t}{4m}, \qquad (r < 2m). \quad (5.144)$$

The Schwarzschild region $0 < r < 2m$ is thus mapped to the region $|u| < v < \sqrt{u^2 + 2m}$ in the (u, v)-plane (Region II in Fig. **??**). The analogue of Eqs. (5.126)–(5.127) are:

$$v^2 - u^2 = (2m - r)e^{r/2m}, \quad (5.145)$$
$$\frac{v}{u} = \tanh\left(\frac{t}{4m}\right). \quad (5.146)$$

The metric components in this region are identical to those of Region I (Eq. (5.133)–(5.135)); the metric can thus again be written as in Eq. (5.136) with the same conformal factor $f^2(u, v)$ (Eq. (5.137)). In Sec. 5.3 (Theorem 5.3.1), we identified $t = -r^* + \text{const.}$ as ingoing photons for $0 < r < 2m$. This corresponds to lines with $u + v = \text{const.}$ in the u-v plane, as in the exterior case. Therefore, in Region I and II, a radially infalling photon moves a long a line with $u + v = \text{const.} > 0$ in direction of increasing coordinate v. Such a photon crosses the line $u = v$ (i.e., $r = 2m$) and falls into the singularity at $r = 0$. For the other ("outgoing") null geodesics with $t = +r^* + \text{const.}$ in Region II, there is no immediate physical interpretation.

Note on conformal transformation. As discussed above, the Kruskal metric is related to the Minkowski metric by a conformal transformation. One can use this fact as a starting point to find the transformation $(r, t) \to (u, v)$ as described in a more 'pedestrian' way in Steps (1)–(4) above. The starting point of Kruskal (1960) was, in fact, to find a coordinate system in which the pathological features of light cones in Schwarzschild coordinates at $r = 2m$ (see Sec. 5.3) are avoided and remain $\mathrm{d}u/\mathrm{d}v = \pm 1$ everywhere on the manifold (as in Minkowski space). These coordinates must therefore arise from the Minkowski coordinates by a conformal (angle-preserving) transformation (Eq. (5.138)). The general form of the metric must then be of the form Eq. (5.136). Imposing this Ansatz (see Eq. (3) in Kruskal 1960), one can easily obtain the transformations Eq. (5.124)–(5.125) and (5.143)–(5.144).

Extension of Schwarzschild: Schwarzschild–Kruskal manifold. One can extend the Schwarzschild spacetime in Kruskal coordinates to the region $v^2 - u^2 < 2m$ with the same metric components (Eq. (5.133)–(5.135)), where the positive radial Schwarzschild coordinate r is related to u and v by (cf. Eqs. (5.126), (5.145))

$$u^2 - v^2 = (r - 2m)e^{r/2m}, \quad |u| \geq |v|, \quad (5.147)$$
$$v^2 - u^2 = (2m - r)e^{r/2m}, \quad |u| \leq |v|. \quad (5.148)$$

This gives rise to the two new regions III and IV in the u-v plane (cf. Fig. **??**). Altogether, Regions I–IV are known as the Schwarzschild-Kruskal manifold; we have thus embedded the

Schwarzschild manifold in this larger Lorentz manifold. In the above derivations, we have assumed the sign of $h(r)$ to be positive; however, these also work assuming a different sign. Assuming a negative sign of $h(r)$ corresponds to a transformation $(u, v) \to (-u, -v)$. Regions I and II are thus isometric to Regions III and IV, respectively. We note a few more characteristics:

- On the entire Kruskal manifold, null geodesics are lines $v = \pm u +$const. and the future light cones are given by tangent vectors $\lambda_u \partial_u + \lambda_v \partial_v$ with $\lambda_v > |\lambda_u|$ (analogous to Minkowski space). The null geodesics $v = u +$ const. in the black hole (Region II) are thus photons that fell into the black hole from Region III.

- Particles or photons cannot enter the region $v < -|u|$ (Region IV)—this region represents a so-called **White Hole**.

- **Causal structure:** Observers in Regions I and III can receive signals from IV and send them into II. Particles and photons that run into II must fall into the singularity at $r = 0$ in finite proper time. Likewise, photons or particles in IV must have originated in the singularity at a previous finite proper time. The singularity in the future (in Region II) is shielded from Regions I and III by an **event horizon**, a causal disconnect at $r = 2m$. Furthermore, there is no causal connection between Regions I and III.

- **Singularities:** As shown here, Kruskal coordinates allow us to 'remove' the singularity of the Schwarzschild coordinates at $r = 2m$. We thus conclude that this is simply a **coordinate singularity**. However, the singularity of the metric along the hyperbola $u^2 - v^2 = -2m$ is a true singularity of the spacetime: the scalar curvature R diverges to infinity here. Since R is invariant (see Sec. 2.11.3), i.e., its value does not depend on the choice of coordinate system, no coordinate system can be found to 'transform this singularity away'. It is thus an actual singularity of the spacetime.

- The Kruskal extension is **maximal**. This means that all geodesics $c : \lambda \mapsto c(\lambda)$ can be either infinitely extended (i.e., defined for all $\lambda \in \mathbb{R}$) or they run into a singularity for a finite value of λ. This spacetime is thus **geodesically incomplete**.[9]

- Another visualization of the Schwarzschild-Kruskal manifold is shown in Fig. **??**, which shows the two-dimensional surface $v = 0$, $\theta = \pi/2$ as a surface of rotation in 3D space. The profile that gives rise to the surface of rotation is given by (cf. Eq. (5.126)):

  $$u^2 = (r - 2m)e^{r/2m}. \tag{5.149}$$

  Note that the upper part of the embedding corresponds to Region I ($u > 0$) and the lower part to Region III ($u < 0$). This is an example of an **Einstein-Rosen bridge**, a connection between *two* otherwise Euclidean spaces, or as the throat of a **wormhole** (also referred to as **Schwarzschild throat**), a connection between two pieces of *one single* Euclidean space, in the limit of large distance between these pieces with respect to the dimensions of the throat.

- The Schwarzschild-Kruskal manifold is static in regions I and III, but it is dynamical in regions II and IV. This is because the Killing field $K = \partial_t$ (not discussed here) becomes spacelike in these regions.

[9] A pseudo-Riemannian manifold is geodesically complete if every maximal geodesic $c : \lambda \mapsto c(\lambda)$ (maximal extension of a local solution of the geodesic equations, cf. Sec. 2.10) is defined on the entire real numbers $\lambda \in \mathbb{R}$.

Chapter 6

Neutron stars

6.1 Stellar structure equations

The equations of stellar structure for non-rotating neutron stars can be obtained from the interior Schwarzschild solution already derived in Sec. 5.1 without much further work. We summarize them in the following

Corollary 6.1.1. ***(Tolman-Oppenheimer-Volkoff equation)***
We shall assume that non-rotating relativistic stars can be modeled by a static isotropic matter distribution as described by an ideal fluid in a spatial region with radius $r \leq R$. The stellar structure equations are then given by

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}r} &= -\frac{Gm\rho}{r^2}\left(1+\frac{p}{\rho c^2}\right)\left(1+\frac{4\pi r^3 p}{mc^2}\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1}, && (6.1)\\ \frac{\mathrm{d}m}{\mathrm{d}r} &= 4\pi r^2\rho, && (6.2)\\ p &= p(\rho, T, \{X_i\}) \qquad \textit{(Equation of state)}, && (6.3) \end{aligned}$$

where p, ρ, T denote pressure, the <u>*total*</u> *mass-energy density*[1]*, temperature, and r denotes the radial coordinate of the interior Schwarzschild solution (cf. Proposition 5.1.1). Furthermore, $m(r)$ denotes the gravitational mass enclosed. The pressure may depend on compositional variables $\{X_i\}$, which define the composition of the stellar material.*

Proof. From energy and momentum conservation, one obtains

$$\begin{aligned} 0 = \nabla_\mu T^{i\mu} &= \frac{1}{\sqrt{|g|}}\partial_\mu\left[\sqrt{|g|}(\rho+p)u^i u^\mu\right] + \Gamma^i_{\mu\nu}(\rho+p)u^\mu u^\nu - \nabla_\mu(pg^{i\mu}) && (6.4)\\ &= \Gamma^i_{00}(\rho+p)u^0u^0 + g^{i\mu}\partial_\mu p, && (6.5) \end{aligned}$$

where we have used the energy-momentum tensor of an ideal fluid (as in Proposition 5.1.1),

$$T_{\mu\nu} = (\rho+p)u_\mu u_\nu + pg_{\mu\nu}, \tag{6.6}$$

[1]As in Proposition 5.1.1, we use ρ here as an abbreviation for the rest-mass density ρ_b plus the internal energy density ϵ, $\rho \equiv \rho_\mathrm{b} + \epsilon$.

the fact that the fluid is at rest, $u^\mu = (u^0, 0, 0, 0)$, and identity (2.249). Setting $i = 1$, we find using the interior Schwarzschild solution,

$$\frac{A'}{2B}(\rho + p)\frac{1}{A} + \frac{p'}{B} = 0 \quad \Leftrightarrow \quad \frac{A'}{2A} = -\frac{p'}{\rho + p}. \tag{6.7}$$

Here, we have made use of the fact that in the Schwarzschild metric, $u^0 = A^{-1/2}$ and

$$\Gamma^1_{00} = -\frac{g^{11}}{2}\partial_1 g_{00} = \frac{A'}{2B} \tag{6.8}$$

(cf. Proposition 5.1.1). Furthermore, from Eqs. (5.33) and (5.36), one also has

$$\frac{rB'}{2B^2} = 4\pi\rho r^2 - \frac{m}{r}. \tag{6.9}$$

Inserting Eqs. (6.7) and (6.9) into Eq. (5.32) yields

$$\frac{rp'}{(\rho + p)B} + \frac{m}{r} = -4\pi p r^2, \tag{6.10}$$

which can be rearranged to give

$$p' = -\frac{m\rho}{r^2}\left(1 + \frac{p}{\rho}\right)\left(1 + \frac{4\pi r^3 p}{m}\right)\left(1 - \frac{2m}{r}\right)^{-1}. \tag{6.11}$$

Equation (6.2) is the differential version of Eq. (5.35). □

Some Remarks.

- Equations (6.1)–(6.2) have been derived and analyzed by Tolman, Oppenheimer, and Volkoff in the 1930s. Equation (6.1) is known as the **Tolman-Oppenheimer-Volkoff (TOV) equation**.

- **Equation of state.** Equations (6.1)–(6.2) represent two coupled ordinary differential equations, which upon integration yield the radial profiles $\rho(r)$, $p(r)$, $m(r)$, etc. that describe the stellar hydrostatic equilibrium. The system of equations in p, ρ, and m is closed by specifying an equation of state (EOS). The EOS expresses dependent thermodynamic variables, such as the pressure, as a function of the independent thermodynamic variables rest-mass density ρ_b, temperature T, and composition $\{X_i\}$. EOSs for neutron star matter may vary dramatically in terms of sophistication, depending on how much physics (realism) is being considered. The most simple ansatz are polytropic EOS of the form

  $$p = p(\rho) = K\rho_\mathrm{b}^\Gamma, \tag{6.12}$$

  where K is the polytropic constant and Γ is the adiabatic constant. For old (cold) neutron stars, temperature has a negligible effect on the pressure, and a polytropic or piecewise polytropic EOS can provide a reasonable approximation to the actual thermodynamic properties.

- **Boundary conditions.** Equations (6.1)–(6.2) can be integrated (from $r = 0$ to $r = R$) given the two boundary conditions

$$m(0) = 0, \qquad p(0) = p_{\rm c} \equiv p(\rho_{\rm b,c}, T_{\rm c}, \{X_{i,\rm c}\}), \tag{6.13}$$

  where "c" refers to the center of the star at $r = 0$. Thus, the only free parameters of the stellar structure equations are those that determine the central pressure.

- **Newtonian limit.** Taking the Newtonian limit of the TOV equation, i.e., using that $p/\rho \ll 1$, $\Phi/c^2 = Gm/rc^2 \ll 1$ (cf. Sec. 4.3), one recovers the equation of hydrostatic equilibrium for Newtonian stars (note that $r^3 p/m \sim p/\rho$):

$$\frac{{\rm d}p}{{\rm d}r} = -\frac{Gm\rho}{r^2}. \tag{6.14}$$

  In order to analyze the deviations from the Newtonian case, let us rewrite Eq. (6.1) in the more suggestive form

$$\frac{{\rm d}p}{{\rm d}r} = -\frac{G(m + 4\pi r^3 p)(\rho + p)}{r^2(1 - 2Gm/rc^2)}. \tag{6.15}$$

  One can thus easily identify the following three modifications with respect to the Newtonian case:

  (i) The enclosed gravitational mass is enhanced by a term proportional to the pressure, since the pressure also acts as a source of the gravitational field (cf. the energy-momentum tensor (6.6)).

  (ii) Gravity also acts on the pressure, so ρ is replaced by $\rho + p$.

  (iii) The additional term in the denominator accounts for the fact that the gravitational force increases faster than $1/r^2$.

We conclude this section with a corollary that follows from the TOV equations:

Corollary 6.1.2. ***(General-relativistic virial theorem)***
The TOV equations imply the following identity:

$$\int (\rho + 3p)\sqrt{|g|}\, d^3x = \int \rho \left(1 - \frac{2m(r)}{r}\right)^{1/2} \sqrt{\gamma}\, d^3x. \tag{6.16}$$

Exercise 6.1.3. (a) *Proof Corollary 6.1.2. Hint: Use integration by parts on the terms on the left-hand side.*

(b) *Show that in the Newtonian limit the general-relativistic virial theorem reduces to the known Newtonian expression,*

$$3\int p\, {\rm d}^3x = -\int \rho \frac{Gm(r)}{r} {\rm d}^3x. \tag{6.17}$$

Exercise 6.1.4. ***(Incompressible star)***

(*a*) *Solve the TOV equations for a star with constant rest-mass density*

$$\rho_\mathrm{b}(r) = \begin{cases} \rho_{\mathrm{b},0} & r \le R \\ 0 & r > R \end{cases} . \tag{6.18}$$

This represents an incompressible fluid, i.e., a fluid for which the pressure is independent of density. We assume that thermal effects can be neglected, i.e., $\rho = \rho_\mathrm{b}$. Show that the pressure profile can be written as

$$p(r) = \rho_{\mathrm{b},0} \frac{\sqrt{1 - \frac{r_\mathrm{S} r^2}{R^3}} - \sqrt{1 - \frac{r_\mathrm{S}}{R}}}{3\sqrt{1 - \frac{r_\mathrm{S}}{R}} - \sqrt{1 - \frac{r_\mathrm{S} r^2}{R^3}}}, \qquad r \le R, \tag{6.19}$$

where $r_\mathrm{S} = 2M$ denotes the Schwarzschild radius. Schwarzschild derived this solution in his second 1916 paper on the Schwarzschild solution[2]

(*b*) *Analyze the central pressure $p(r = 0)$ to show that such stars are only stable provided that*

$$R > \frac{9}{8} r_\mathrm{S} = \frac{9M}{4}. \tag{6.20}$$

This ***stability criterion*** *provides an upper limit for the mass, given a stellar radius R, or, equivalently a lower limit for the stellar radius, given a fixed mass M. Note that this is a fundamental GR effect and that it does not depend on matter properties. If not satisfied, the matter distribution will undergo* ***gravitational collapse****.*

6.2 Gravitational versus baryonic mass

Outside the star (i.e., the matter distribution), spacetime is given by the exterior Schwarzschild solution, which is determined by the **gravitational mass** (cf. Corollary 6.1.1, Proposition 5.1.1)

$$M = 4\pi \int_0^R \rho r^2 \mathrm{d}r. \tag{6.21}$$

We note that r refers to the Schwarzschild radial coordinate—integration here is not carried out in a flat Euclidean space—, and that we integrate over the total mass-energy density $\rho = \rho_\mathrm{b} + \epsilon$, where ρ_b is the rest-mass density and ϵ is the internal energy density. The gravitational mass is different from the **baryonic mass**,

$$M_\mathrm{b} = N m_\mathrm{b} = \int_{\mathrm{star},\, t=\mathrm{const.}} \rho_\mathrm{b} \sqrt{\gamma} \mathrm{d}^3 x = 4\pi \int_0^R \frac{\rho_\mathrm{b} r^2}{\sqrt{1 - 2m(r)/r}} \mathrm{d}r, \tag{6.22}$$

where N is the total number of nucleons (baryons) of the star, m_b the mass per baryon, and γ the determinant of the spatial part of the Schwarzschild metric $g_{\mu\nu}$ (cf. Proposition 5.1.1), $\gamma = \det(g_{ij})$. The above integral is computed on a fixed time slice $t =$ const. in Schwarzschild coordinates, and $\sqrt{\gamma}$ represents the corresponding spatial volume element.

[2] K. Schwarzschild, "Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie", Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Proceedings of the Prussian Academy of Sciences) **7**, 424, February 24, 1916.

The total internal energy, which corresponds to the **gravitational binding energy** (also known as **mass defect**), is given by

$$E/c^2 \equiv \Delta M = M_\mathrm{b} - M = E_\mathrm{grav}/c^2. \tag{6.23}$$

One can decompose E as

$$E = -(T + V), \tag{6.24}$$

where

$$T \equiv 4\pi \int_0^R \frac{\epsilon(r) r^2}{\sqrt{1 - 2m(r)/r}} \mathrm{d}r \tag{6.25}$$

and

$$V \equiv 4\pi \int_0^R \left[\rho(r) r^2 \left(1 - \frac{1}{\sqrt{1 - 2m(r)/r}}\right)\right] \mathrm{d}r. \tag{6.26}$$

Expanding the square roots in the above integrals,

$$T = 4\pi \int_0^R \epsilon(r) r^2 \left(1 + \frac{m(r)}{r} + \ldots\right) \mathrm{d}r \tag{6.27}$$

$$V = -4\pi \int_0^R \rho(r) r^2 \left(\frac{m(r)}{r} + \frac{3}{2}\frac{m^2(r)}{r^2} + \ldots\right) \mathrm{d}r \tag{6.28}$$

shows that in the Newtonian limit T and V correspond to the Newtonian internal and gravitational energy of the star, respectively.

Chapter 7

Weak Gravitational Fields

7.1 Linearized field equations

As already noted in Sec. 4.3, for most astrophysical systems (except for compact objects) gravitational fields are weak, and the so-called linearized theory is applicable to a good approximation. We return to a more detailed discussion of this limit here. As in Sec. 4.3 we shall assume that there exist local charts (U, ϕ) with coordinates (x^0, x^1, x^2, x^3) in which the metric components can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1. \tag{7.1}$$

The so-called **linearized theory** of GR results from expanding Einstein's field equations with $\Lambda = 0$ using Eq. (7.1) and assuming that $|h_{\mu\nu}|$ is sufficiently small, such that only linear terms in the perturbations $h, \partial_\nu h$, etc., need be retained. Retaining only terms up to linear order in the perturbations implies that indices are raised and lowered with $\eta_{\mu\nu}$; in particular, this procedure is equivalent to considering the spacetime (7.1) as a "gravitational field $h_{\mu\nu}$" on a flat background spacetime $\eta_{\mu\nu}$, similar to the electromagnetic field A^μ in electrodynamics.

In linearized theory, the components of the Ricci tensor (cf. Def. 2.11.10) are given by

$$R_{\mu\nu} \simeq \partial_l \Gamma^l_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\lambda\mu}, \tag{7.2}$$

and the Christoffel symbols (cf. Eq. (2.135)) read

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}(h^\alpha{}_{\mu,\nu} + h^\alpha{}_{\nu,\mu} - h_{\mu,\nu}{}^{;\alpha}). \tag{7.3}$$

This results in the following expressions for the Ricci tensor and the scalar curvature

$$R_{\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h^\sigma{}_\sigma) \tag{7.4}$$

$$R = \partial_\sigma\partial_\rho h^{\sigma\rho} - \Box h, \tag{7.5}$$

where $\Box \equiv \partial^\sigma\partial_\sigma$ denotes the d'Alembertian in linearized theory[1] and where we defined the trace of $h_{\mu\nu}$:

$$h \equiv \eta^{\mu\nu} h_{\mu\nu}. \tag{7.6}$$

[1]Covariant derivatives acting on $h_{\mu\nu}$ are reduced to ordinary partial derivatives in linearized theory, since the Christoffel symbols in linearized theory, $\Gamma^\rho_{\mu\nu} = (\eta^{\rho\sigma}/2)(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu})$, are already first order in $h_{\mu\nu}$, $\nabla_\nu h^{\alpha\beta} = \partial_\nu h^{\alpha\beta} + \Gamma^\alpha_{\nu\sigma} h^{\sigma\beta} + \Gamma^\beta_{\nu\sigma} h^{\alpha\sigma} = \partial_\nu h^{\alpha\beta}$.

For further reference, we also define

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{7.7}$$

Inserting these expressions into Einstein's field equations (4.2) and rewriting in terms of $\bar{h}_{\mu\nu}$, one finds the **linearized field equations**

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu\bar{h}_{\mu\rho} - \partial^\rho\partial_\mu\bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\mu\nu}. \tag{7.8}$$

Coordinate transformations. In suitable coordinates (a 'suitable gauge'), the linearized Einstein equations reduce to a simpler expression than (7.8). Consider a coordinate diffeomorphism

$$\begin{array}{llll} \psi: & V = \phi(U) \subset \mathbb{R}^4 & \to & V' \subseteq \mathbb{R}^4 \\ & x^\mu & \mapsto & x'^\mu(x) \equiv x^\mu + \xi^\mu(x) \end{array}, \qquad |\partial_\alpha\xi_\beta| \leq |h_{\mu\nu}|, \tag{7.9}$$

where $\xi^\mu(x)$ is a differentiable vector field on V; then $\psi \circ \phi$ is also a chart of U defining new coordinates x'^μ on U. According to the general transformation law for tensors under arbitrary coordinate changes (cf. Theorem (2.4.5)), $g_{\mu\nu}$ transforms as

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}g'_{\rho\sigma}(x'). \tag{7.10}$$

For the coordinate diffeomorphism (7.9), $\partial x'^\sigma/\partial x^\mu = \delta^\sigma_\mu + \partial_\mu\xi^\sigma$ and thus we obtain up to first order in small quantities

$$g_{\mu\nu}(x) = g'_{\mu\nu}(x') + g'_{\mu\sigma}(x')\partial_\nu\xi^\sigma + g'_{\nu\sigma}(x')\partial_\mu\xi^\sigma, \tag{7.11}$$

which translates into

$$h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu) \tag{7.12}$$

using Eq. (7.1). Alternatively, in terms of $\bar{h}_{\mu\nu}$ Eq. (7.12) is written as

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\sigma\xi^\sigma). \tag{7.13}$$

The above considerations immediately yield the following important result:

Theorem 7.1.1. *In the linearized theory of GR the Riemann tensor, with components given by*

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\mu\partial_\sigma h_{\nu\rho} + \partial_\nu\partial_\rho h_{\mu\sigma} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}), \tag{7.14}$$

is invariant, rather than just covariant—that is, the Riemann tensor is invariant under coordinate transformations of the type (7.9).

Exercise 7.1.2. *Proof Theorem 7.1.1.*

Let us now focus again on discussing the linearized field equations (7.8). The terms that appear on the left-hand side of this equation suggest the following

Definition 7.1.3 (Lorentz gauge). *Let $\bar{h}_{\mu\nu}$ be defined as in* (7.7). *The Lorentz gauge (also called the Hilbert or harmonic gauge) is defined by*

$$\boxed{\partial^\nu\bar{h}_{\mu\nu} = 0.} \tag{7.15}$$

Equation (7.15) can always be achieved by applying a suitable coordinate transformation of the type (7.9). Let $f_\mu(x) \equiv \partial^\nu \bar{h}_{\mu\nu}$. Observe that, according to Equation (7.13),

$$(\partial^\nu \bar{h}_{\mu\nu})' = \partial^\nu \bar{h}_{\mu\nu} - \Box\xi_\mu. \tag{7.16}$$

Defining $\xi^\mu(x)$ by $\Box\xi_\mu(x) = f_\mu(x)$, i.e.,

$$\xi_\mu(x) = \int \mathrm{d}^4 y\, G(x-y) f_\mu(y), \tag{7.17}$$

where $G(x)$ is the Green's function of the d'Alembertian, $\Box G(x) = \delta(x)$, we have found coordinates x'^μ on U, such that Equation (7.15) is satisfied.

In Lorentz gauge, the linearized field equations simplify considerably and reduce to a wave equation,

$$\boxed{\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}.} \tag{7.18}$$

As discussed in Sec. 1.2, Einstein's first attempt to generalize Newton's theory of gravitation within the framework of special relativity suggested a linear field equation of this type. The result obtained here shows that indeed, for weak gravitational fields, such linear field equations are obtained. All non-linearities of the theory that arise via the mass-energy equivalence are removed. This is reflected in the fact that energy-momentum conservation in linearized theory reduces to the special relativistic version: applying the Lorentz gauge conditions (7.15) to Equation (7.18) yields

$$\partial_\nu T^{\mu\nu} = 0. \tag{7.19}$$

This implies that the gravitational field generated by $T_{\mu\nu}$ does not react back on the source.

The most general solution to the linearized field equations (7.18), subject to the gauge condition (7.15), is a sum of a particular (retarded) solution and the solution of the corresponding homogeneous wave equation. The wave solutions to the homogeneous equation are called **gravitational waves**. We shall discuss such solutions in Chapter 8; they represent propagating perturbations in spacetime. The retarded solution can be obtained with the help of the **retarded Green's function** $G(x)$ of the d'Alembertian operator,

$$\Box G(x-x') = \delta(x-x'), \tag{7.20}$$

given by

$$G(x-x') = -\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\delta[(x^0 - |\boldsymbol{x}-\boldsymbol{x}'|) - x^{0\prime}]; \tag{7.21}$$

the result is

$$\bar{h}_{\mu\nu}(x) = -16\pi G \int \mathrm{d}^4 x'\, G(x-x') T_{\mu\nu}(x') \tag{7.22}$$

$$= 4G \int \mathrm{d}^3 x'\, \frac{T_{\mu\nu}(x^0 - |\boldsymbol{x}-\boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{7.23}$$

Finally, we note that if $\bar{h}_{\mu\nu}$ is known from Eq. (7.18) (or, alternatively, from Eq. (7.8)), $h_{\mu\nu}$ can easily be obtained with the help of Equation (7.7),

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}, \tag{7.24}$$

where $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = h - 2h = -h$ was used.

7.2 Nearly Newtonian fields

Using the formalism set up in the previous section, we shall now investigate the nearly Newtonian limit of general relativity in more detail. The analysis conducted here adds to the discussion in Sec. 4.3 and provides relations that are useful for many astrophysical systems, such as, e.g., planetary systems, and for some classic tests of general relativity.

For nearly Newtonian sources, $T_{00} \gg |T_{0j}|, |T_{ij}|$, as T_{0j} are linear and $T_{ij}$are quadratic in the velocities (cf. Eq. (1.17)). Furthermore, we assume velocities to be so small that retardation effects of the slowly varying source can be neglected. Then Eq. (7.23) reduces to

$$\bar{h}_{00}(x) = 4G \int \mathrm{d}^3x' \, \frac{T_{00}(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} = -4\Phi(t, \mathbf{x}), \quad \bar{h}_{0j} = \bar{h}_{ij} = 0, \tag{7.25}$$

where $\Phi(t, \mathbf{x})$ is the Newtonian gravitational potential (cf. Eq. (1.6)). Inserting this into Eq. (7.24) and Eq. (7.1), one finds for the metric:

$$g_{00} = -(1 + 2\Phi), \;\; g_{0i} = 0, \;\; g_{ij} = (1 - 2\Phi)\delta_{ij}. \tag{7.26}$$

The corresponding line element thus reads

$$\mathrm{d}s^2 = -(1 + 2\Phi)\mathrm{d}t^2 + (1 - 2\Phi)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{7.27}$$

At large distances, the monopole contribution to the Newtonian gravitational potential dominates, and one obtains

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 + \frac{2m}{r}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{7.28}$$

The spatial part of the metric is thus non-Euclidean even for nearly Newtonian fields. This is in contrast to what one may have concluded from Eq. (4.18) in Sec. 4.3, namely that $g_{00} = -(1 + \Phi)$ and $g_{ij} = \delta_{ij}$, $g_{0j} = 0$. Essentially, the latter form was used by Einstein in his 1911 paper to compute the deflection of light.[2] The non-Euclidean nature of the spatial part of the metric accounts for the famous factor of two by which his preliminary 1911 result was inaccurate compared to the correct GR result (see Sec. 5.2.3).

[2] A. Einstein, "On the Influence of Gravitation on the Propagation of Light", Annalen der Physik 35 (1911), 898-908. See also: The Collected Papers of Albert Einstein, Vol. 3, Doc. 23, https://einsteinpapers.press.princeton.edu/papers

Chapter 8

Gravitational Waves

In this chapter, we shall discuss wave solutions of the linearized Einstein equations, which constitute propagating perturbations of spacetime—gravitational waves (GWs). The starting point are the linearized field equations in Lorentz gauge (Eq. (7.18)). The expression on the right-hand side of that equation is the source function, which, under conditions to be specified later (see Section 8.3), generates gravitational radiation.

When GWs are produced by a system according to Eq. (7.18), i.e., within the framework of linearized theory, matter described by $T_{\mu\nu}$ is assumed to move in flat spacetime and energy-momentum conservation is expressed by the special-relativistic expression Eq. (7.19). In this case, GWs are the only source of curvature and the dynamics of self-gravitating systems, such as a (binary) star, is described by Newtonian physics. This is inevitable, since otherwise GWs interact with the gravitational field produced by the matter content in a dynamical, (highly) non-linear way[1]; however, by definition of the linearized theory, non-linearities cannot be accounted for. For astrophysical objects with weak gravitational fields according to Eq. (7.1) (see also Sec. 4.3), generation of GWs by, e.g., binary systems composed of such objects is described with sufficient accuracy within linearized theory for our purposes (cf. Sections 8.4).

We first discuss GW solutions to the vacuum equations in Secs. 8.1 and 8.2, discuss the generation of such waves (Sec. 8.3), and apply the previous results to astronomical binary systems (Sec. 8.4).

8.1 Vacuum equations

We shall first discuss wave solutions through vacuum, i.e., the propagation of GWs outside the source once they have been generated. Outside the source, $T_{\mu\nu} = 0$ and Eq. (7.18) reduces to the linearized field equations in vacuo,

$$\Box \bar{h}_{\mu\nu} = 0. \tag{8.1}$$

These equations show that **GWs propagate with the speed of light** ($\Box = -(1/c^2)\partial_t^2 + \boldsymbol{\nabla}^2$). If $T_{\mu\nu} = 0$, an even more suitable coordinate frame than the one defined by the Lorentz gauge can be found, such that Equation (8.1) is further simplified.

[1] In order to include the gravitational field of the source, higher-than-linear-order corrections to flat spacetime are required, such that one may want to employ, e.g., a so-called post-Newtonian expansion of Einstein's field equations.

The Lorentz gauge (7.15) imposes four conditions on the components $\bar{h}_{\mu\nu}$ and thus it reduces the 10 degrees of freedom of $h_{\mu\nu}$ (symmetric tensor) to six independent components. However, a second, subsequent coordinate transformation of the type (7.9) with $\Box\xi^\mu = 0$ can be applied, which does not destroy the Lorentz gauge (cf. Equation (7.16)). The components ξ^μ can be fixed such as to further simplify Equation (8.1), thereby reducing the number of independent components of $\bar{h}_{\mu\nu}$ to just two degrees of freedom. From Equation (7.13) we recall that $\bar{h}_{\mu\nu}$ is changed by $\xi_{\mu\nu} \equiv \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\sigma\xi^\sigma$ under a coordinate transformation of the type (7.9). Since $\Box\xi^\mu = 0$, $\Box\xi_{\mu\nu} = 0$ and thus $\bar{h}'_{\mu\nu}$ still satisfies Equation (8.1). Let, for example, $\xi^0(x)$ be chosen such that $\bar{h}' = 0$, which yields $\bar{h}'_{\mu\nu} = h'_{\mu\nu}$. Furthermore, the functions $\xi^i(x)$ can be chosen such that $\bar{h}'_{0i} = h'_{0i} = 0$. It is important to note that the latter conditions cannot be imposed inside the source, since there, $\Box\bar{h}'_{\mu\nu} \neq 0$, i.e., individual components of $\bar{h}_{\mu\nu}$ cannot be set to zero. With the aforementioned conditions, the Lorentz gauge condition for $\mu = 0$ reads $\partial^0 h'_{00} = 0$; hence, as far as the time-dependent part of a solution to Equation (8.1) is concerned, that is, as far as GWs are concerned, we can set $h'_{00} = 0$. We summarize these observations in form of the following

Theorem and Definition 8.1.1 (**TT gauge**)**.** *There is a coordinate system in which the time-dependent part of a solution $h_{\mu\nu}$ to the linearized Einstein equations* (7.8) *in vacuo satisfies the gauge conditions*

$$\boxed{h_{0\mu} = 0, \quad h^i{}_i = 0, \quad \partial^j h_{ij} = 0.} \tag{8.2}$$

These conditions define the so-called transverse-traceless (TT) gauge[2] *and they reduce the 10 degrees of freedom of $h_{\mu\nu}$ to only two independent components.*

In TT gauge, the linearized Einstein equations in vacuo reduce to

$$\boxed{\Box h_{ij} = 0.} \tag{8.3}$$

It can be shown that it is impossible to find a coordinate frame in which the number of independent components of $h_{\mu\nu}$ can be further reduced, i.e., GWs cannot be "gauged away"; the remaining two degrees of freedom give rise to the two independent polarization states of a GW (see Section 8.2). This is also evident from a field theoretical approach to linearized theory via the **Pauli-Fierz action** (see, e.g., Maggiore 2007, Chapter 2). In this context, $h_{\mu\nu}$ is treated as a classical field in flat spacetime and it is easily shown that $h_{\mu\nu}$ is a massless spin-2 field—that is, the quanta of this field, the **gravitons**, are massless particles with spin $s = 2$. This means that the graviton is a massless representation of the Poincaré group, the representation being two dimensional if the theory under consideration conserves parity (like gravity or electromagnetism); the two degrees of freedom are the two possible helicity states $\pm s$, which are interpreted as two independent polarization states of the particle.

In the following section, some important properties of GWs are discussed, which follow directly or indirectly from Equation (8.3). However before proceeding to this, and for completeness, we briefly discuss the GW energy-momentum tensor and the general definition of GWs in presence of a curved background spacetime.

Energy-momentum tensor of GWs. Inserting the metric (7.1) into Def. 2.11.10, we expand the Ricci tensor up to second order in $h_{\mu\nu}$,

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + O(h^3), \tag{8.4}$$

[2] Motivation for the term "transverse" is given in Section 8.2.

where $R^{(0)}_{\mu\nu} = 0$, since Minkowski space is flat. When inserted into Einstein's field equations (4.2) without a cosmological constant, the second-order terms can be grouped together,

$$R^{(2)}_{\mu\nu} - \frac{1}{2}(Rg_{\mu\nu})^{(2)} \equiv -8\pi G\, t_{\mu\nu}, \tag{8.5}$$

such that Einstein's field equations up to second order in $h_{\mu\nu}$ read

$$G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}R^{(1)}\eta_{\mu\nu} = 8\pi G\,(T_{\mu\nu} + t_{\mu\nu}). \tag{8.6}$$

By construction, $\nabla_\nu G^{\mu\nu} = 0$ (cf. Sec. 4.1) and thus, $\nabla^\nu G^{(1)}_{\mu\nu} = 0$. In particular, this implies that $\partial^\nu G^{(1)}_{\mu\nu} = 0$ in linearized theory. This can also be seen from Equation (7.19) together with the left-hand side of Equation (7.8), which equals $-2G^{(1)}_{\mu\nu}$. Consequently, Eqns. (8.6) and (7.19) yield

$$\partial^\nu t_{\mu\nu} = 0, \tag{8.7}$$

which together with Eqn. (8.6) suggest that $t_{\mu\nu}$ is to be interpreted as the **energy-momentum tensor of GWs**. Evaluating the left-hand side of Equation (8.5) for $h_{\mu\nu}$ in TT gauge one obtains the explicit expression (see, e.g., Maggiore 2007)

$$\boxed{t_{\mu\nu} = \frac{1}{32\pi G}\langle \partial_\mu h_{ij} \partial_\nu h^{ij} \rangle,} \tag{8.8}$$

where we already applied a suitable temporal average $\langle\rangle$. It is important to note that this expression contains only the contributions to the energy-momentum tensor that cannot be gauged away and that are therefore inherent to the GW field (see the discussion below Theorem and Definition 8.1.1).

General definition of GWs. We have only considered the linearized theory so far, in which GWs are defined as small fluctuations around a flat background spacetime. However, in the most general case of a curved, dynamical spacetime, such as, for example, in the vicinity of a black hole binary, it is a priori not clear that propagating perturbations (GWs) on a "background spacetime" exist, i.e., that an unambiguous splitting

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x), \qquad |h_{\mu\nu}| \ll |\bar{g}_{\mu\nu}(x)|, \tag{8.9}$$

exists between a dynamical curved background described by $\bar{g}_{\mu\nu}(x)$ and some perturbations $h_{\mu\nu}$ upon it. Such a splitting is possible if there is a coordinate system such that $g_{\mu\nu}$ can be separated into either (a) a low-frequency background and a high-frequency perturbation or (b) a spatially slowly-varying background and small ripples on it (so-called *short-wave approximation*). The following theorem provides a more general definition of GWs, which contains the definition of GWs in linearized theory as the special case in which $\bar{g}_{\mu\nu}(x) = \eta_{\mu\nu}$ (for a more detailed discussion, see, e.g., Maggiore 2007).

Theorem and Definition 8.1.2 (**GWs in curved spacetime**). *Given the separation of scales (a) or (b) mentioned above and assuming that the background metric $\bar{g}_{\mu\nu}$ dominates the curvature, Einstein's field equations split into a field equation for the background metric,*

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} \simeq \frac{8\pi G}{c^4}\bar{T}_{\mu\nu}, \tag{8.10}$$

and a wave equation for the perturbations $h_{\mu\nu}$ outside the matter sources,

$$\Box \bar{h}_{\mu\nu} = 0, \tag{8.11}$$

where $\bar{h}_{\mu\nu}$ is in Lorentz gauge, $\bar{\nabla}^\nu \bar{h}_{\mu\nu} = 0$. Here, $\Box \equiv \bar{\nabla}^\sigma \bar{\nabla}_\sigma$, and $\bar{\nabla}_\mu$ is the covariant derivative with respect to the background metric. Furthermore, $\bar{h}_{\mu\nu}$ is obtained from $h_{\mu\nu}$ as in Eq. (7.7) and, depending on the cases (a) or (b), the bar over $R_{\mu\nu}$, R, and T represents a suitable temporal or spatial average[3]*. For obvious reasons, the perturbations $h_{\mu\nu}$ are called gravitational waves.*

8.2 Some properties of the GW tensor in TT gauge

This section briefly summarizes some important properties of the GW tensor in TT gauge. In particular, Eq. (8.3) is solved and the polarization of GWs is addressed. Furthermore, given a GW tensor in Lorentz gauge, we discuss how to transform it into TT gauge.

Spherical components. In TT gauge (cf. Def. 8.1.1), the GW tensor satisfies the linearized field equations in vacuo of the form (8.3). Since h_{ij} is traceless and symmetric (spin-2 operator), it can be expanded as

$$h_{ij} = \sum_{m=-2}^{2} h_m \mathcal{Y}_{ij}^{2m}, \tag{8.12}$$

where the expansion coefficients h_m are called the five independent **spherical components** of h_{ij} and where the expansion was made in terms of the following basis for the five-dimensional space of traceless symmetric tensors:

$$\mathcal{Y}_{ij}^{2,\pm 2} = \sqrt{\frac{15}{32\pi}} \begin{pmatrix} 1 & \pm \mathrm{i} & 0 \\ \pm \mathrm{i} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}, \tag{8.13}$$

$$\mathcal{Y}_{ij}^{2,\pm 1} = \mp\sqrt{\frac{15}{32\pi}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \pm \mathrm{i} \\ 1 & \pm \mathrm{i} & 0 \end{pmatrix}_{ij}, \tag{8.14}$$

$$\mathcal{Y}_{ij}^{20} = \sqrt{\frac{5}{16\pi}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}_{ij}. \tag{8.15}$$

These tensors are related to spherical harmonics by the following relation:

$$Y_{2m}(\theta, \phi) = \mathcal{Y}_{ij}^{2m} n^i n^j, \tag{8.16}$$

where n_i denote the components of the radial unit vector

$$\boldsymbol{n} \equiv \boldsymbol{e}_r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{8.17}$$

Furthermore, we mention the orthogonality relation that applies to these tensors:

$$\sum_{ij} \mathcal{Y}_{ij}^{2m} \left(\mathcal{Y}_{ij}^{2m'}\right)^* = \frac{15}{8\pi} \delta^{mm'}. \tag{8.18}$$

[3]Equation (8.10) is obtained from Einstein's equation by applying a renormalization group transformation.

With the help of this orthogonality relation, Eq. (8.12) can be inverted to give an explicit expression for the spherical components:

$$\boxed{h_m = \frac{8\pi}{15} h^{ij} \left(\mathcal{Y}_{ij}^{2m}\right)^* .} \tag{8.19}$$

Moreover, multiplying Equation (8.12) by $n_i n_j$, summing over i and j, and inserting Eq. (8.16), we obtain

$$h_{ij} n^i n^j = \sum_{m=-2}^{2} h_m Y_{2m}(\theta, \phi). \tag{8.20}$$

Furthermore, inverting Eq. (8.16) with the help of Eq. (8.18) yields

$$n_i n_j - \frac{1}{3}\delta_{ij} = \sum_{m=-2}^{2} c_{ij}^m \, Y_{2m}(\theta, \phi), \tag{8.21}$$

where $c_{ij}^m = \frac{8\pi}{15}(\mathcal{Y}_{ij}^{2m})^*$. The factor of 1/3 on the left-hand side of Eq. (8.21) is fixed by the requirement that the left hand side be traceless.

Plane wave expansion and polarization. We shall now derive a convenient expression for the general solution to Eq. (8.3). Since Eq. (8.3) is a simple wave equation, its general solution is given by a superposition of plane waves,

$$h_{ij}(x) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k \left[\mathcal{A}_{ij}(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}kx} + \mathcal{A}_{ij}^*(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}kx}\right], \tag{8.22}$$

where, as usual, $x = (t, \boldsymbol{x})$ denotes spacetime points and where $k^\mu \equiv (\omega, \boldsymbol{k})$ is the usual four-wave vector. The integral in equation (8.22) can alternatively be expressed as an integral over frequency $\nu = \omega/2\pi$ and solid angle Ω,

$$h_{ij}(x) = \int_0^\infty \mathrm{d}\nu \, \nu^2 \int \mathrm{d}\Omega \left[\mathcal{A}_{ij}(\nu, \boldsymbol{n}) \mathrm{e}^{-\mathrm{i}2\pi\nu(t - \boldsymbol{n}\cdot\boldsymbol{x})} + c.c.\right], \tag{8.23}$$

since $\mathrm{d}^3 k = |\boldsymbol{k}|^2 \mathrm{d}|\boldsymbol{k}| \mathrm{d}\Omega = (2\pi)^3 \nu^2 \mathrm{d}\nu \mathrm{d}\Omega$. Here $\boldsymbol{n}$ was used to denote the propagation direction of the individual plane waves, $\boldsymbol{n} = \boldsymbol{k}/|\boldsymbol{k}|$.

The gauge condition $\partial^j h_{ij} = 0$ (cf. Eq. (8.2)) applied to Eqs. (8.22) or (8.23) yields

$$k^j \mathcal{A}_{ij}(\boldsymbol{k}) = n^j \mathcal{A}_{ij}(\nu, \boldsymbol{n}) = 0. \tag{8.24}$$

Whence it follows that an individual plane wave propagating in the direction of $\boldsymbol{n}$ or a superposition of plane waves with the same propagation direction $\boldsymbol{n}$, $h_{ij}^{\boldsymbol{n}}$, satisfy $n^j h_{ij}^{\boldsymbol{n}} = 0$. Consequently, the GW tensor $h_{ij}^{\boldsymbol{n}}$ is *transverse* with respect to the direction of propagation, which motivates the term "transverse" in Def. 8.1.1. Furthermore, from Eq. (8.2) it follows that $\mathcal{A}^i{}_i = 0$, i.e., $\mathcal{A}_{ij}$ has only two degrees of freedom, which define the **polarization** of $h_{ij}^{\boldsymbol{n}}$.

We now define the symmetric **polarization tensors**

$$e_{ij}^{+}(\boldsymbol{n}) \equiv \boldsymbol{u}_i \boldsymbol{u}_j - \boldsymbol{v}_i \boldsymbol{v}_j, \tag{8.25}$$

$$e_{ij}^{\times}(\boldsymbol{n}) \equiv \boldsymbol{u}_i \boldsymbol{v}_j + \boldsymbol{u}_j \boldsymbol{v}_i, \tag{8.26}$$

where $\boldsymbol{n}$ given by Eq. (8.17), $\boldsymbol{u}$, and $\boldsymbol{v}$ are pairwise orthonormal vectors,

$$\boldsymbol{u} \equiv \boldsymbol{e}_\Theta = (\cos\Theta\cos\phi, \cos\Theta\sin\phi, -\sin\Theta), \tag{8.27}$$

$$\boldsymbol{v} \equiv \boldsymbol{e}_\phi = (-\sin\phi, \cos\phi, 0). \tag{8.28}$$

From this definition, it is clear that the polarization tensors are transverse with respect to $\boldsymbol{n}$, i.e., $n^j e^+_{ij}(\boldsymbol{n}) = n^j e^\times_{ij}(\boldsymbol{n}) = 0$. Moreover, they are linearly independent,

$$e^A_{ij}(\boldsymbol{n}) e^{A',ij}(\boldsymbol{n}) = 2\delta^{A,A'}, \tag{8.29}$$

which can be proved explicitly by inserting the definitions (8.25) and (8.26). Here $A = \{+, \times\}$ labels the so-called **polarization states**. Due to the fact that $\mathcal{A}_{ij}$ is transverse with respect to $\boldsymbol{n}$, cf. Eq. (8.24), it can be expanded in terms of the polarization tensors,

$$\mathcal{A}_{ij}(\nu, \boldsymbol{n}) = \frac{1}{\nu^2} \sum_A h_A(\nu, \boldsymbol{n}) e^A_{ij}(\boldsymbol{n}), \tag{8.30}$$

where the expansion coefficients $h_A(\nu, \boldsymbol{n})$ are called the **Fourier amplitudes** of h_{ij}. Substituting Eq. (8.30) into Eq. (8.23) and extending the frequency integration domain to minus infinity by setting

$$h_A(-\nu, \boldsymbol{n}) \equiv h^*_A(\nu, \boldsymbol{n}) \tag{8.31}$$

finally yields

$$\boxed{h_{ij}(t, \boldsymbol{x}) = \sum_A \int_{-\infty}^{\infty} \mathrm{d}\nu \int \mathrm{d}\Omega\, h_A(\nu, \boldsymbol{n}) e^A_{ij}(\boldsymbol{n}) \mathrm{e}^{-\mathrm{i}2\pi\nu(t - \boldsymbol{n}\cdot\boldsymbol{x})}.} \tag{8.32}$$

This is the desired form of the plane wave expansion for the GW tensor.

As a trivial corollary of the above derivation, a GW propagating in direction of $\boldsymbol{n}$ can be written as

$$h^{\boldsymbol{n}}_{ij}(x) = h_+(x) e^+_{ij}(\boldsymbol{n}) + h_\times(x) e^\times_{ij}(\boldsymbol{n}), \tag{8.33}$$

where h_+ and $h_\times$ are called the **waveforms** of the GW,

$$h_+(x) = \int_{-\infty}^{\infty} \mathrm{d}\nu\, h_+(\nu, \boldsymbol{n}) \mathrm{e}^{-\mathrm{i}2\pi\nu(t - \boldsymbol{n}\cdot\boldsymbol{x}/c)} \tag{8.34}$$

$$h_\times(x) = \int_{-\infty}^{\infty} \mathrm{d}\nu\, h_\times(\nu, \boldsymbol{n}) \mathrm{e}^{-\mathrm{i}2\pi\nu(t - \boldsymbol{n}\cdot\boldsymbol{x}/c)}. \tag{8.35}$$

Hence, in general, such a GW is a superposition of the two independent polarization states $A = \{+, \times\}$, with the waveforms specifying the respective **polarization amplitudes**[4]. The naming conventions "+" (plus) and "×" (cross) are motivated by the action of a GW on test masses: consider a ring of test masses located in, e.g., the (x, y) plane and an entirely plus or cross polarized GW propagating along the z axis. Under the influence of the GW the ring of test masses is elliptically deformed either with the principle axes of the instantaneous ellipse along the x and y axes at all times (entirely plus polarized) or with the principle axes along a coordinate system (x', y') that is obtained from the (x, y) system by a 45° rotation (entirely cross polarized).

[4]It can easily be shown that the combinations $h_\times \mp h_+$ define the helicity eigenstates with helicities ± 2, respectively (cf. also the note below Theorem 8.1.1)

Projection onto the TT gauge. Given a GW tensor in Lorentz gauge, one can obtain the GW tensor in TT gauge with respect to a propagation direction $\boldsymbol{n} = \boldsymbol{x}/|\boldsymbol{x}|$ in a straightforward way. This is achieved with the help of the **Lambda tensor**,

$$\Lambda_{ij,kl}(\boldsymbol{x}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}, \tag{8.36}$$

where

$$P_{ij}(\boldsymbol{x}) \equiv \delta_{ij} - n_i n_j \tag{8.37}$$

is a symmetric, transverse projection tensor with trace $P^i_{\ i} = 2$, i.e., $n^i P_{ij}(\boldsymbol{x}) = 0$ and $P_{ik}P_{kj} = P_{ij}$. The Lambda tensor is transverse with respect to all indices, it is traceless with respect to (i, j) and (k, l), and it has the projector property

$$\Lambda_{ij,kl}\Lambda^{kl,}{}_{mn} = \Lambda_{ij,mn}. \tag{8.38}$$

Moreover, we mention the explicit form of the Lambda tensor in terms of the propagation direction $\boldsymbol{n}$ for further reference,

$$\begin{aligned}\Lambda_{ij,kl}(\boldsymbol{x}) &= \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} \\ &\quad + \frac{1}{2}n_k n_l \delta_{ij} + \frac{1}{2}n_i n_j \delta_{kl} + \frac{1}{2}n_i n_j n_k n_l.\end{aligned} \tag{8.39}$$

Let $h_{\mu\nu}$ denote the GW tensor in Lorentz gauge (7.15) satisfying the linearized field equations in vacuo (8.1). Then, the GW tensor in TT gauge with respect to $\boldsymbol{n}$, h^{TT}_{ij}, is obtained by the following operation:

$$\boxed{h^{TT}_{ij} = \Lambda_{ij,kl}h_{kl}.} \tag{8.40}$$

We note that the right-hand side is transverse and traceless by construction and that it satisfies Eq. (8.3).

8.3 Generation of GWs in linearized theory

In this section, we discuss the generation of gravitational radiation in the framework of linearized theory. The formalism presented here sets the scene for calculating the emission of GWs by important astrophysical sources, such as, for example, inspiralling binary systems (Sec. 8.4) and cosmological sources that produce a stochastic background of GW radiation (not discussed here).

As mentioned at the beginning of this chapter, matter that sources GWs according to Eq. (7.18) is assumed to move in flat spacetime in linearized theory—that is, GWs are the only source of curvature and the dynamics of self-gravitating systems, such as a (binary) star, are described by Newtonian physics. This obviously requires that the typical velocities v inside a self-gravitating system be small compared to the speed of light, $v \ll c$. In contrast, if the dynamics of a system are determined by non-gravitational forces, the laws of special relativity apply and the weak field expansion of linearized theory becomes independent of the typical velocity inside the system. In this section, we explicitly retain the speed of light in all formulae, rather than setting $c = 1$; this is instructive in discussing some of the approximations made here.

The linearized Einstein Equations (7.18) for a radiation problem are solved with the help of the *retarded Green's function* of the d'Alembertian operator as discussed in Sec. 7.1. Here, we are interested in the GW signal exterior to the source at large (astronomical) distances $r \gg R$, where R denotes the typical radius of the source. In this case,

$$|\boldsymbol{x} - \boldsymbol{x}'| = r - \frac{\boldsymbol{x}' \cdot \boldsymbol{x}}{r} + O\left(\frac{R^2}{r}\right). \tag{8.41}$$

This leads to

$$\boxed{h_{ij}(t, \boldsymbol{x}) = \frac{1}{r}\frac{4G}{c^2}\Lambda_{ij,kl}(\boldsymbol{x}) \int \mathrm{d}^3x'\, T^{kl}\left(t - \frac{r}{c} + \frac{\boldsymbol{x}' \cdot \boldsymbol{x}}{rc}, \boldsymbol{x}'\right),} \tag{8.42}$$

where we have already projected $\bar{h}_{\mu\nu}$ from Equation (7.23) onto the TT gauge according to Equation (8.40). Equation (8.42) specifies the gravitational radiation that is emitted in the direction of $\boldsymbol{x}$.

Low-velocity expansion. As stated above, it is now assumed that the typical velocity inside the source is small compared to the speed of light, $v \ll c$. Let ω_s denote the typical frequency of motions inside the source; then $v \sim \omega_s R$ and thus $\omega_s R \ll c$. Moreover, the frequency of the emitted GW radiation, ω, is of the same order than ω_s (to be justified later); hence, the Fourier transform $\hat{T}_{kl}(\omega, \boldsymbol{k})$ of $T_{kl}(t, \boldsymbol{x})$ peaks around ω_s, which yields

$$\frac{\omega}{c}\frac{\boldsymbol{x}' \cdot \boldsymbol{x}}{r} \lesssim \frac{\omega_s R}{c} \ll 1. \tag{8.43}$$

Expressing T_{kl} in terms of its Fourier transform, we have the following expansion:

$$\begin{aligned}
T^{kl}\left(t - \frac{r}{c} + \frac{\boldsymbol{x}' \cdot \boldsymbol{x}}{rc}, \boldsymbol{x}'\right) &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \hat{T}^{kl}(\omega, \boldsymbol{k}) \mathrm{e}^{-\mathrm{i}\omega(t - r/c + \boldsymbol{x}' \cdot \boldsymbol{x}/rc) + \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}'} \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \hat{T}^{kl}(\omega, \boldsymbol{k}) \mathrm{e}^{-\mathrm{i}\omega(t - r/c) + \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}'} \\
&\quad \times \left[1 - i\frac{\omega}{c}\frac{x'^i x_i}{r} + \frac{1}{2!}\left(-i\frac{\omega}{c}\right)^2 \frac{x'^i x_i x'^j x_j}{r^2} + \dots\right] \\
&= \left[T^{kl} + \frac{x'^i x_i}{rc}\partial_t T^{kl} + \frac{1}{2!}\frac{x'^i x_i x'^j x_j}{r^2 c^2}\partial_t^2 T^{kl} + \dots\right]_{\mathrm{ret}},
\end{aligned}$$

where "ret" indicates that the quantities T^{kl}, $\partial_t T^{kl}$, $\partial_t^2 T^{kl}$, etc., are evaluated at the retarded time $t_{\mathrm{ret}} \equiv t - r/c$. Defining the **momenta of** T^{ij},

$$S^{ij}(t) \equiv \int \mathrm{d}^3x\, T^{ij}(t, \boldsymbol{x}), \tag{8.44}$$

$$S^{ij,a_1 \dots a_n}(t) \equiv \int \mathrm{d}^3x\, T^{ij}(t, \boldsymbol{x}) x^{a_1} \dots x^{a_n}, \tag{8.45}$$

and inserting the above expansion into Eq. (8.42), we arrive at the following expansion for the GW tensor in TT gauge:

$$\boxed{h_{ij}(t, \boldsymbol{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\boldsymbol{x})\left[S^{kl} + \frac{x_a}{cr}\dot{S}^{kl,a} + \frac{1}{2!}\frac{x_a x_b}{c^2 r^2}\ddot{S}^{kl,ab} + \dots\right]_{\mathrm{ret}},} \tag{8.46}$$

where the dots above the quantities $S^{kl,a}$, $S^{kl,ab}$, etc., denote derivatives with respect to time. It is important to note that this is a systematic expansion in powers of v/c.

As already stated at the beginning of this section, there are two cases to be distinguished. If the source is a self-gravitating system, $v \ll c$ is required for this formalism to be applicable. In this case, the first term in the expansion (8.46) will dominate and an expansion to higher orders is not necessary, since higher-order corrections must include effects of curvature due to the matter distribution of the source, which is not possible in linearized theory. If, however, the dynamics of the source are governed by non-gravitational forces, the expansion can be carried out to arbitrarily high order, since the velocities inside the system are allowed to reach arbitrarily high values; even the case $v \lesssim c$ can be considered by solving the exact expression in v/c (Eq. (8.42)). As the most interesting astrophysical sources for GWs are self-gravitating systems and since we are particularly interested in binary systems here, we shall solely analyze the leading term in Eq. (8.46) below.

Dimensionally, T^{00}/c^2 is a mass density and therefore, we define in analogy to Eqs .(8.44) and (8.45) the **mass moments** of the source by

$$M(t) \equiv \frac{1}{c^2}\int \mathrm{d}^3x\, T^{00}(t,\boldsymbol{x}), \tag{8.47}$$

$$M^{a_1\ldots a_n}(t) \equiv \frac{1}{c^2}\int \mathrm{d}^3x\, T^{00}(t,\boldsymbol{x})x^{a_1}\ldots x^{a_n}. \tag{8.48}$$

This definition leads to the following

Theorem 8.3.1. *In linearized theory,*

$$S^{ij} = \frac{1}{2}\ddot{M}^{ij} \tag{8.49}$$

for a source that has finite extent.

Proof. From energy-momentum conservation in linearized theory (cf. Eq. (7.19)) it follows that

$$\begin{aligned} 0 &= \int \mathrm{d}^3x\,(\partial_\nu T^{\mu\nu})x^k = \frac{1}{c}\partial_t\int \mathrm{d}^3x\, T^{\mu 0}x^k + \int \mathrm{d}^3x\,(\partial_l T^{\mu l})x^k \\ &\overset{\text{p.I.}}{=} \frac{1}{c}\partial_t\int \mathrm{d}^3x\, T^{\mu 0}x^k - \int \mathrm{d}^3x\, T^{\mu k}, \end{aligned} \tag{8.50}$$

where we used the fact that $T^{\mu\nu}$ vanishes at infinity. Likewise, employing Gauss' theorem,

$$\frac{1}{c}\partial_t\int \mathrm{d}^3x\, T^{00}x^kx^l = \int \mathrm{d}^3x\,\partial_\nu(T^{0\nu}x^kx^l). \tag{8.51}$$

Consequently,

$$\begin{aligned} \frac{1}{2}\ddot{M}^{ij} &= \frac{1}{2c^2}\partial_t^2\int \mathrm{d}^3x\, T^{00}x^ix^j \overset{(8.51),(7.19)}{=} \frac{1}{2c}\partial_t\int \mathrm{d}^3x\, T^{0\nu}\partial_\nu(x^ix^j) \\ &= \frac{1}{2c}\partial_t\int \mathrm{d}^3x\,(T^{0i}x^j + T^{0j}x^i) \overset{(8.50)}{=} \int \mathrm{d}^3x\, T^{ij} = S^{ij} \end{aligned}$$

□

The symmetric mass moment M^{ij} can be decomposed into its traceless and trace parts as

$$M^{kl} = \left(M^{kl} - \frac{1}{3}\delta^{kl} M^m{}_m \right) + \frac{1}{3}\delta^{kl} M^m{}_m, \tag{8.52}$$

where the trace part vanishes when contracted with the Lambda tensor $\Lambda_{ij,kl}$, since the latter is traceless in (k, l). Defining the **quadrupole tensor**,

$$Q^{ij}(t) \equiv M^{ij} - \frac{1}{3}\delta^{ij} M^m{}_m = \int \mathrm{d}^3x\, \rho(t, \boldsymbol{x}) \left(x^i x^j - \frac{1}{3} r^2 \delta^{ij} \right), \tag{8.53}$$

where $\rho = T^{00}/c^2$ is the mass density[5], and employing Theorem 8.3.1, Eq. (8.46) in the case of a self-gravitating system is rewritten as

$$\boxed{h_{ij}(t, \boldsymbol{x}) = \frac{1}{r}\frac{2G}{c^4} \Lambda_{ij,kl}(\boldsymbol{x}) \ddot{Q}^{kl}(t - r/c).} \tag{8.54}$$

Alongside the corresponding expression for the total power emitted by the source (gravitational luminosity), which we shall derive below, this equation is often referred to as the so-called **quadrupole formula**. Equation (8.54) shows that the lowest order of the multipole expansion (8.46) is quadrupole radiation. The **absence of gravitational monopole and dipole radiation** is due to the fact that the graviton as a massless particle with helicity ± 2 (cf. the note below Theorem 8.1.1) cannot be put into a state with total angular momentum $j < 2$.

Finally, we calculate the energy flux of the GW field at sufficiently large distances r from the source, i.e., where the field can be described by the quadrupole approximation (8.54). Let E_V denote the energy of the GW field inside a spherical shell of volume V centred on the GW source, with its inner boundary S_1 and its outer boundary S_2 at large distances from the source. Then,

$$\frac{1}{c}\frac{\mathrm{d}E_V}{\mathrm{d}t} = \partial_0 \int_V \mathrm{d}^3x\, t^{00} \overset{(8.7)}{=} -\int_V \mathrm{d}^3x\, \partial_i t^{0i} = -\int_{\partial V} \mathrm{d}S\, n_i t^{0i}, \tag{8.55}$$

where $t^{\mu\nu}$ denotes the energy-momentum tensor of the GWs (cf. Eq. (8.8)) and $\boldsymbol{n}$ is the outer unit normal to ∂V, that is, $\boldsymbol{n} = -\boldsymbol{e}_r$ on S_1 and $\boldsymbol{n} = \boldsymbol{e}_r$ on S_2. The surface element is denoted by $\mathrm{d}S = r^2 \mathrm{d}\Omega$. Setting $\boldsymbol{n} = \boldsymbol{e}_r$, we know from Eq. (8.8) that

$$n_i t^{0i} = \frac{c^4}{32\pi G} \langle \partial^0 h_{ab} n_i \partial^i h^{ab} \rangle = \frac{c^4}{32\pi G} \langle \partial^0 h_{ab} \frac{\partial}{\partial r} h^{ab} \rangle, \tag{8.56}$$

where h_{ij} is given by Eq. (8.54) and is evaluated at the retarded time. Hence, using $\frac{\partial}{\partial r} h_{ij}(t - r/c, \boldsymbol{x}) = \partial^0 h_{ij}(t - r/c, \boldsymbol{x}) + O(1/r^2)$,

$$\frac{1}{c}\frac{\mathrm{d}E_V}{\mathrm{d}t} = \int_{S_1} \mathrm{d}S_1\, t^{00} - \int_{S_2} \mathrm{d}S_2\, t^{00}, \tag{8.57}$$

which is the difference between the incoming energy through S_1 and the energy flowing out through S_2. We are solely interested in the energy flux that flows outward through S_2,

$$\frac{\mathrm{d}E}{\mathrm{d}S\mathrm{d}t} = c t^{00} = \frac{c^3}{32\pi G} \langle \dot{h}_{ij} \dot{h}^{ij} \rangle, \tag{8.58}$$

[5] To lowest order in v/c, $\rho = T^{00}/c^2$ is dominated by the mass density.

which we can identify with the energy flux carried by the GWs. For a GW field of the form (8.33), this reads

$$\frac{\mathrm{d}E}{\mathrm{d}S\mathrm{d}t} = \frac{c^3}{32\pi G}\langle\sum_{A,A'}\dot{h}_A\dot{h}_{A'}\boldsymbol{e}^A_{ij}\boldsymbol{e}^{A',ij}\rangle \overset{(8.29)}{=} \frac{c^3}{16\pi G}\langle\dot{h}_+^2+\dot{h}_\times^2\rangle. \tag{8.59}$$

In the quadrupole approximation (8.54), the total radiated power (**gravitational luminosity**) is given by

$$P = \int \mathrm{d}\Omega\,\frac{\mathrm{d}P}{\mathrm{d}\Omega} \overset{(8.58)}{=} \frac{c^3 r^2}{32\pi G}\int \mathrm{d}\Omega\,\langle\dot{h}_{ij}\dot{h}^{ij}\rangle \overset{(8.36),(8.38)}{=} \frac{G}{8\pi c^5}\langle\dddot{Q}^{ij}\dddot{Q}^{kl}\rangle\int\mathrm{d}\Omega\,\Lambda_{ij,kl}, \tag{8.60}$$

which leads to the famous **quadrupole formula**.

$$\boxed{P = \frac{G}{5c^5}\langle\dddot{Q}_{ij}\dddot{Q}^{ij}\rangle,} \tag{8.61}$$

Here, we have made use of the identity

$$\int\mathrm{d}\Omega\,\Lambda_{ij,kl} = \frac{2\pi}{15}(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \tag{8.62}$$

and the fact that Q^{ij} is traceless and symmetric.

8.4 GWs from inspiralling binary systems

In this section, we discuss the generation of gravitational waves from inspiralling binary systems in linearized theory, i.e., based on the quadrupole formula (8.54), which is sufficiently accurate to roughly estimate the signal strengths of astronomical sources at large distances. For many decades this has been a rather academic exercise; however, since the first direct detection of gravitational waves from a binary black hole merger with heavy black holes of mass $\sim 30\,M_\odot$ by the gravitational-wave observatory LIGO Abbott et al. (2016), the detection of gravitational waves has become reality and is developing into a routine business. Accurately predicting waveforms of GWs for merging binary systems has become an important field of gravitational-wave science. This is because the detector noise is typically orders of magnitude larger than the weak astrophysical signal, so that the latter has to be "dug out" of the noise by so-called matched filtering techniques. In order to achieve a reasonable chance for detection of astrophysical systems and to accurately estimate the source parameters (so-called "parameter estimation" analysis), more precise computations than linearized theory, typically up to high order in the post-Newtonian expansion, are required.

The evolution of a binary system composed of neutron stars (NSs) or black holes (BHs) is characterized by a long *inspiral phase*, during which the binary system gradually spirals inwards due to the emission of GWs[6], the subsequent *merger*, when the stars coalesce, and the final *ringdown phase*, during which the resulting excited object (a neutron star or black

[6] This has first been observed and interpreted by Hulse and Taylor (Hulse & Taylor 1975; Taylor et al. 1979; Taylor & Weisberg 1982), for which they were awarded the Nobel Prize in 1993. They observed that the decrease in the orbital period of the binary was in agreement with the emission of gravitational waves as predicted from linearized theory.

hole) emits GWs to radiate away the energy stored in normal mode oscillations. Whereas the ringdown phase can be described by perturbation theory of the Kerr spacetime, the highly dynamical non-linear merger process itself is particularly difficult to model; essentially, it is only accessible via numerical relativity simulations. Here, however, we are solely interested in gravitational radiation from the inspiral phase. For most of the inspiral phase, the signal is quasi-monochromatic (i.e., with only a small frequency drift) and it is *universal* in the sense that the GW signal does not depend on the detailed nature of the objects (they can be assumed to be point-like particles).

Apart from the known black hole and neutron star binary mergers that LIGO and Virgo have detected so far, there are other known sources with well defined strain amplitudes. These are double white dwarf (DWD) systems, the most abundant galactic binaries with a population of $\sim 3 \times 10^7$ objects (**???**), and AM Canum Venaticorum (AM CVn) systems (white dwarfs accreting helium-rich material from a compact companion; **?**). The final evolutionary stages of these types of binary systems are still poorly known; they are considered as potential progenitors of type Ia supernovae (**???**). As in the case of neutron star and black hole binaries, the GW signal is quasi-monochromatic and universal before the final spiral-in phase. Some known galactic AM CVn systems with well known gravitational strain amplitudes (accurately measured distances through parallaxes) serve as 'verification binaries' for the future Laser Interferometer Space Antenna (LISA) mission[7].

Circular binaries, i.e., binaries on circular orbits, are of particular importance, since eccentric orbits are usually circularized. Initial eccentric orbits are circularized by tidal interactions between the binary components before DWD binaries are born (**???**); AM CVn binaries are also commonly thought to have circular orbits (**????**). Eccentric orbits are often attributed to neutron-star and black-hole binaries. Such initial eccentricity may arise because of 'kicks' received during supernova explosions and/or because of dynamical binary formation scenarios (as opposed to isolated binary evolution scenarios) in dynamical environments such as globular clusters or nuclear star clusters. However, orbits are often circularized long before the merger due to the backreaction of the emitted GWs on the orbit of the binary.

8.4.1 Circular binaries

The above discussion leads us to consider the inspiral of a circular binary and to compute the waveforms in linearized theory, that is, using Eqn. (8.54). Since the dynamics of the source must be described by Newtonian physics (cf. Secs. 7.1 and 8.3), the component stars can be described as point masses m_1, m_2 at positions $\boldsymbol{r}_1$, $\boldsymbol{r}_2$. In the center of mass (CM) frame, this reduces to an effective one-body problem with mass $\mu = m_1 m_2/(m_1 + m_2)$, $\ddot{\boldsymbol{r}} = -(Gm/r^3)\boldsymbol{r}$, where $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ and $m = m_1 + m_2$. The *orbital type* of a binary (elliptic, circular) is defined by the orbit that is traced out by the reduced mass in the CM frame. Let us consider a circular binary in the (x, y) plane,

$$x_{\rm o}^1(t) = r_{\rm o} \cos\left(\omega_s t + \frac{\pi}{2}\right), \tag{8.63}$$

$$x_{\rm o}^2(t) = r_{\rm o} \sin\left(\omega_s t + \frac{\pi}{2}\right), \tag{8.64}$$

$$x_{\rm o}^3(t) = 0, \tag{8.65}$$

[7]The *Laser Interferometer Space Antenna* (*LISA*) is a full-sky GW observatory in the frequency range $10^{-4} - 0.1$ Hz to be launched in 2034 (**??**).

where

$$\omega_s^2 = \frac{Gm}{r_\text{o}^3} \tag{8.66}$$

is the orbital frequency and r_o denotes the orbital radius; the initial phase of $\pi/2$ was chosen for convenience. Given the mass density of $\rho(t, \boldsymbol{x}) = \mu\delta(\boldsymbol{x} - \boldsymbol{r}_\text{o}(t))$, we know from Eqs. (8.53) and (8.54) that

$$h_{ij}(t, \boldsymbol{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\boldsymbol{x})\ddot{M}^{kl}(t) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\boldsymbol{x})\mu x_\text{o}^i(t)x_\text{o}^j(t), \tag{8.67}$$

where the non-vanishing mass moments are given by

$$M_{11} = \mu r_\text{o}^2 \frac{1 - \cos 2\omega_s t}{2}, \tag{8.68}$$

$$M_{22} = \mu r_\text{o}^2 \frac{1 + \cos 2\omega_s t}{2}, \tag{8.69}$$

$$M_{12} = -\frac{1}{2}\mu r_\text{o}^2 \sin 2\omega_s t. \tag{8.70}$$

Plugging in the explicit expression (8.39) for the Lambda tensor, performing the contraction with $\ddot{M}^{kl}$, and decomposing h_{ij} according to Eq. (8.33) yields

$$h_+(t, \boldsymbol{x}) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\omega_s}{c}\right)^{2/3}\frac{1 + \cos^2\theta}{2}\cos(2\omega_s t_\text{ret} + 2\phi), \tag{8.71a}$$

$$h_\times(t, \boldsymbol{x}) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\omega_s}{c}\right)^{2/3}\cos\theta\sin(2\omega_s t_\text{ret} + 2\phi), \tag{8.71b}$$

where, as usual, $t_\text{ret} = t - r/c$, $\boldsymbol{x} = \boldsymbol{x}(r, \theta, \phi)$, and M_c is the **chirp mass**,

$$M_c \equiv \mu^{3/5}m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}. \tag{8.72}$$

Alternatively, one can write

$$h_+(t, \boldsymbol{x}) = \frac{4G}{rc^4}\mu r_\text{o}^2\omega_s^2\frac{1 + \cos^2\theta}{2}\cos(2\omega_s t_\text{ret} + 2\phi), \tag{8.73a}$$

$$h_\times(t, \boldsymbol{x}) = \frac{4G}{rc^4}\mu r_\text{o}^2\omega_s^2\cos\theta\sin(2\omega_s t_\text{ret} + 2\phi). \tag{8.73b}$$

From Eqs. (8.71) and (8.73) one can see that the **frequency ν_gw of the GWs emitted is twice the orbital frequency** of the source, $\nu_\text{gw} = 2\nu_s = \omega_s/\pi$. A distant observer would naturally define a coordinate system (x', y', z') such that the z' axis is aligned with the line of sight (LOS). Then, θ is equal to the *inclination* ι of the binary, which is the angle between the normal to the orbit (z axis) and the LOS (z' axis). During a typical observation, the distance r to the source can be assumed to be constant and the proper motion of the source can be neglected, such that $\alpha \equiv 2(\phi - \omega_s r/c)$ is a constant. The gravitational radiation from a binary as seen by a distant observer can then be written as

$$\boxed{\begin{aligned} h_+(t) &= \frac{4}{rc^4}(GM_c)^{5/3}(\pi\nu_\text{gw})^{2/3}\frac{1 + \cos^2\iota}{2}\cos(2\pi\nu_\text{gw}t + \alpha), \quad (8.74\text{a}) \\ h_\times(t) &= \frac{4}{rc^4}(GM_c)^{5/3}(\pi\nu_\text{gw})^{2/3}\cos\iota\sin(2\pi\nu_\text{gw}t + \alpha). \quad (8.74\text{b}) \end{aligned}}$$

Table 8.1: Predicted GW strain amplitudes h and frequencies $\nu_{\rm gw}$ of five AM CVn binary systems, the so-called LISA verification binaries (**?**).

Star	$\nu_{\rm gw}$ (mHz)	h
AM CVn	1.944	$2.0^{+0.4}_{-0.3} \times 10^{-22}$
HP Lib	1.813	$3.7^{+0.6}_{-0.8} \times 10^{-22}$
CR Boo	1.360	$2.1^{+0.4}_{-0.5} \times 10^{-22}$
V803 Cen	1.253	$3.0^{+0.5}_{-0.7} \times 10^{-22}$
GP Com	0.7158	$[4.0 - 6.6] \times 10^{-23}$

As is evident from Eqs. (8.74), gravitational radiation from a binary system is in general **elliptically polarized**, that is, in the $(h_+, h_\times)$ plane the radiation traces out an ellipse parametrized by t. If the LOS is perpendicular to the orbital plane of the binary ($\iota = 0$), the radiation is **circularly polarized**, and if the LOS lies in the orbital plane ($\iota = \pi/2$), the observer detects a **linearly plus polarized** GW. Therefore, the inclination ι of the binary orbit can be obtained from a measurement of the relative amplitude of h_+ and $h_\times$. Such a measurement lifts the **distance–inclination degeneracy** present in Eq. (8.74) if only one of the components can be accurately measured. In order to obtain precise distance measurements of a gravitational-wave source, it is therefore important that a network of gravitational-wave interferometers is sensitive to both polarization amplitudes. The arms of the two LIGO detectors have been built with the exact same orientation; this helps to maximize coincidence and thus to make detections more easily, but it minimizes the sensitivity to both polarization amplitudes.

Another important point to realize from Eqs. (8.74) is the fact that the amplitude of GWs decays with distance (luminosity distance) as $\propto r^{-1}$. This is in contrast to electromagnetic waves, which decay as $\propto r^{-2}$. As the sensitivity volume of the LIGO/Virgo/Kagra network of gravitational-wave interferometers grows to cosmological volumes, it will thus become increasingly difficult to identify electromagnetic counterparts. For example, optical or near-infrared counterparts such as kilonovae (thermal radiation that results from the radioactive decay of heavy isotopes synthesized by the rapid neutron capture process in neutron star mergers; Metzger et al. 2010; Metzger 2017), observed in GW170817, the first neutron-star merger detected by LIGO/Virgo (Abbott et al. 2017a,c; **?**), will only be observable throughout the LIGO/Virgo design sensitivity volume for neutron star mergers with the help of new facilities such as the Large Synoptic Survey Telescope (LSST) and the James Webb Space Telescope (JWST).

Table 8.1 lists the so-called LISA verification binaries. They are circular binaries considered to guarantee the detection of GWs of known amplitude; they can thus be used to test the detector (**?**). Here, the intrinsic GW strain amplitude h is defined by

$$h = \left[\frac{1}{2}\left(|h_+|^2 + |h_\times|^2\right)\right]^{1/2}, \tag{8.75}$$

where h_+ and $h_\times$ denote the respective waveform amplitudes (**???**). These verification binaries have strain amplitudes of the order of $h \sim 10^{-22}$.

8.4.2 Quasi-circular binaries.

Gravitational waves emitted by a circular binary remove energy and angular momentum from the binary system. As we shall show now, this energy and angular momentum drain reacts back onto the orbit of the binary, that is, it shrinks the orbit. A gradual change of the orbit, in turn, gradually changes the GW signal in a characteristic way—it causes the signal to 'chirp'. We shall now investigate this process and derive the waveforms of such a quasi-circular binary.

The power radiated in GWs by a circular binary is

$$P_{\rm GW} = \int \mathrm{d}\Omega \, \frac{\mathrm{d}P_{\rm GW}}{\mathrm{d}\Omega} \overset{(8.59)}{=} \frac{c^3 r^2}{16\pi G} \int \mathrm{d}\Omega \, \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \tag{8.76a}$$

$$\overset{(8.71)}{=} \frac{2c^5}{\pi G} \left(\frac{G M_c \omega_{\rm gw}}{2c^3} \right)^{10/3} \int \mathrm{d}\Omega \, g(\theta) \tag{8.76b}$$

$$= \frac{32}{5} \frac{c^5}{G} \left(\frac{G M_c \omega_{\rm gw}}{2c^3} \right)^{10/3}, \tag{8.76c}$$

where

$$g(\theta) \equiv \left(\frac{1 + \cos^2\theta}{2} \right)^2 + \cos^2\theta. \tag{8.77}$$

The energy of the binary (cf. the viral theorem),

$$E = E_{\rm kin} + E_{\rm pot} = -\frac{G m_1 m_2}{2 r_{\rm o}} \overset{(8.66)}{=} -\left(\frac{G^2 M_c^5 \omega_{\rm gw}^2}{32} \right)^{1/3}, \tag{8.78}$$

is reduced due to the radiated power

$$P = -\frac{\mathrm{d}E}{\mathrm{d}t}. \tag{8.79}$$

According to Eq. (8.78), the orbital radius must decrease, which, in turn, causes the orbital frequency to increase (cf. Eq. (8.66)). However, since the emitted power increases with orbital frequency according to Equation (8.76c), $r_{\rm o}$ must further decrease, which eventually leads to the coalescence of the stars at a certain time $t_{\rm coal}$. Here, we are interested in the regime in which the variation of the orbital radius is much smaller than the orbital velocity, $|\dot{r}_{\rm o}| \ll \omega_s r_{\rm o}$, which translates into

$$\dot{\omega}_s \ll \omega_s^2, \tag{8.80}$$

using Eq. (8.66). The orbit is *quasi-circular*, with a slowly varying orbital radius, the aforementioned equations remain valid to first order, and thus the emitted gravitational radiation is *quasi-monochromatic*.

Inserting Eqs. (8.76c) and (8.78), Eq. (8.79) can be written as

$$\dot{\nu}_{\rm GW} = \frac{96}{5} \pi^{8/3} \left(\frac{G M_c}{c^3} \right)^{5/3} \nu_{\rm GW}^{11/3}. \tag{8.81}$$

Integrating this expression one obtains

$$\nu_{\rm GW}(\tau) = \frac{1}{\pi} \left(\frac{5}{256} \frac{1}{\tau} \right)^{3/8} \left(\frac{G M_c}{c^3} \right)^{-5/8}. \tag{8.82}$$

Here $\tau \equiv t_{\rm coal} - t$ is the time to coalescence. Let

$$\Phi(\tau) \equiv 2\pi \int_{\tau}^{\tau_0} \mathrm{d}\tau' \, \nu_{\rm GW}(\tau') \tag{8.83a}$$

$$\stackrel{(8.82)}{=} -2\left(\frac{5GM_c}{c^3}\right)^{-5/8} \tau^{5/8} + \Phi_0, \tag{8.83b}$$

denote the absolute phase of the binary's motion, where we have set $\tau_0 = \tau(t=0) = 0$ and $\Phi_0 = \Phi(\tau = 0)$ in the second line. As long as Eq. (8.80) is satisfied, the GW amplitude as seen by a distant observer can be obtained from Eqs. (8.74) by replacing $\nu_{\rm GW}$ in front of the trigonometric functions by $\nu_{\rm GW}(t)$ (Eq. (8.82)) and substituting the argument $2\pi\nu_{\rm GW}t + \alpha$ of the trigonometric functions with $\Phi(\tau)$[8]:

$$\boxed{\begin{aligned} h_+(t) &= \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c\tau}\right)^{1/4} \frac{1+\cos^2\iota}{2}\cos[\Phi(\tau)], \quad (8.84a)\\ h_\times(t) &= \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c\tau}\right)^{1/4} \cos\iota\sin[\Phi(\tau)]. \quad (8.84b)\end{aligned}}$$

The above equations show a **chirping behavior**, that is, both the amplitude and the frequency of the GW increase as the objects approach coalescence. Note that the amplitude increases $\propto \tau^{-1/4} \stackrel{\tau\to 0}{\to} \infty$ and that the frequency increases (cf. Eq. (8.82)) $\nu_{\rm GW} \propto \tau^{-3/8} \stackrel{\tau\to 0}{\to} \infty$.

8.4.3 Quasi-circular binaries in frequency space

In practice, in the context of GW detection one typically requires expressions of the GW signal in frequency space. The computation of the Fourier transforms of Eqs. (8.84) is complicated by the fact that h_+ and $h_\times$ are formally only defined in the range $-\infty < t < t_{\rm coal}$. We solely cite the result here and refer to Maggiore (2007) Problem 4.1 for a derivation:

$$\boxed{\begin{aligned} \tilde{h}_+(f) &= \left(\frac{5}{24\pi^{4/3}}\right)^{1/2} e^{i\Psi_+(f)}\frac{c}{r}\left(\frac{GM_c}{c^3}\right)^{5/6}\frac{1}{f^{7/6}}\frac{1+\cos^2\iota}{2}, \quad (8.85a)\\ \tilde{h}_\times(f) &= \left(\frac{5}{24\pi^{4/3}}\right)^{1/2} e^{i\Psi_\times(f)}\frac{c}{r}\left(\frac{GM_c}{c^3}\right)^{5/6}\frac{1}{f^{7/6}}\cos\iota. \quad (8.85b)\end{aligned}}$$

The phases are given by

$$\Psi_+(f) = 2\pi f(t_{\rm coal} + r/c) - \Phi_0 - \frac{\pi}{4} + \frac{3}{4}\left(\frac{GM_c}{c^3}8\pi f\right)^{-5/3}, \tag{8.86}$$

$$\Psi_\times(f) = \Psi_+(f) + \frac{\pi}{2}, \tag{8.87}$$

where $\Phi_0 = \Phi(\tau = 0)$ is the phase Φ in the time domain at coalescence, as defined above (cf. Eq. (8.83)).

[8] If Eq. (8.80) is not satisfied, derivatives of $r_{\rm o}$ and ω_s with respect to time must be included in the derivation leading to Eqs. (8.74). However, since the innermost stable circular orbit (ISCO) of the Schwarzschild geometry is normally reached before Eq. (8.80) breaks down, one can assume Eq. (8.80) to hold essentially throughout the inspiral phase at least up to first order.

Merger frequency (maximum inspiral frequency). The above derivation of waveforms is based on the Newtonian effective one-body point-mass problem. In this Newtonian context, the orbital radius r_o of the effective reduced mass (distance between the two point masses) can become arbitrarily small. However, in a realistic astrophysical scenario, as the two compact objects approach, curvature effects of spacetime due to the strong gravitational fields of the objects become important and start to affect the inspiral dynamics. Assuming one of the objects is much heavier than the other, the inspiral scenario is essentially that of a test mass in a Schwarzschild spacetime of mass m. Stable circular orbits do not exist in Schwarzschild spacetime interior to the so-called Innermost Stable Circular Orbit (ISCO). This radius essentially determines the end of the inspiral phase, after which a 'catastrophic' plunge and merger will occur. For systems with similar masses (and if the objects have spin) there are corrections to this picture; however, the ISCO of a Schwarzschild spacetime corresponding to the total mass $m = m_1 + m_2$ of the system provides a reasonable estimate for the minimum inspiral separation of two point masses,

$$r_{\mathrm{ISCO}} = \frac{6Gm}{c^2}. \tag{8.88}$$

The corresponding maximum inspiral frequency $\nu_{\max}$ can be estimated from the source frequency ν_{s} at a separation r_{ISCO} using the Keplerian expression (to be consistent with the Newtonian setting above),

$$\nu_{\max} \simeq \nu_{\mathrm{s}}(r_{\mathrm{ISCO}}) = \left(\frac{Gm}{4\pi^2 r_{\mathrm{ISCO}}^3}\right)^{1/2} = \frac{1}{12\pi\sqrt{6}}\frac{c^3}{Gm} \simeq 2.2\,\mathrm{kHz}\left(\frac{m}{M_\odot}\right)^{-1}. \tag{8.89}$$

Notice that this merger frequency only depends on the total mass of the binary system, and that it scales linearly inverse with that mass $m = m_1 + m_2$.

From the above expression, one can directly estimate that a typical binary neutron star merger (two neutron stars, each with typical mass $m_1 \simeq m_2 \simeq 1.4M_\odot$), merges at $\nu_{\max} \simeq 800\,\mathrm{Hz}$, while a binary stellar-mass black hole system with typical masses $m = 10M_\odot$ merges at $\nu_{\max} \simeq 200\,\mathrm{Hz}$. The LIGO-Virgo-Kagra gravitational-wave detectors are designed with a sensitivity regime of $\sim 10 - 1000\,\mathrm{Hz}$, precisely to detect the final part of the inspiral and coalescence of these compact objects.

Merging galaxies are binary black hole systems composed of supermassive black holes with $m > 10^6 M_\odot$. This shifts the merger frequencies into the mHz regime and below. The LISA mission will be able to detect such mergers in the mHz to μHz regime, while pulsar timing arrays may soon be able to detect binary black hole mergers in the nHz regime.

8.4.4 Quasi-circular binaries in the cosmological context

The above derivations have assumed a flat background spacetime in which gravitational waves propagate from the source to the observer. However, gravitational-wave sources may reside at large distances (high redshift z) and thus gravitational waves are affected by cosmological expansion as they travel through the Universe to the observer. A detailed analysis of propagation of GWs in a homogeneous and isotropic Universe (a Friedmann-Robertson-Walker (FRW) cosmology) shows that all expressions for waveforms in the observer frame derived above formally remain valid, both in time and frequency space, provided the following substitutions are made (see, e.g., Maggiore (2007), Sec. 4.1.4):

$$r \rightarrow d_L, \tag{8.90}$$
$$M_c \rightarrow M_z \equiv (1+z)M_c, \tag{8.91}$$
$$f \rightarrow f_z \equiv f/(1+z). \tag{8.92}$$

Here, d_L denotes the luminosity distance, which is defined by

$$F = \frac{L}{4\pi d_L^2} \tag{8.93}$$

for a source of electromagnetic waves with intrinsic radiated luminosity L (power per unit time) and flux F (energy per unit time per unit area) received by an observer. In an FRW cosmology, the luminosity distance as a function of redshift z is given by

$$d_L(z) = (1+z)\int_0^z \frac{c\,dz'}{H(z')}, \tag{8.94}$$

where $H(z)$ is the Hubble parameter, with $H_0 = H(0)$ being the Hubble constant. Notice that the product of chirp mass and frequency is conserved, $M_c f = M_z f_z$, and that the waveform expressions apply to any FRW background, i.e., they are independent of any specific cosmology used. In fact, observations of such GWs allow us to measure cosmological parameters; one example is discussed below.

8.4.5 Standard sirens

As first pointed out by Schutz (1986), GW observations offer the exciting possibility to measure the luminosity distance to the source if h_+, $h_\times$, and $\dot\nu_{GW}$ can simultaneously be measured from a chirping binary signal. Since h_+ and $h_\times$ have the same amplitude factors (in particular, the same dependence on chirp mass), but different dependence on the inclination, the inclination $\cos\iota$ can directly be inferred from a measurement of $h_+/h_\times$ (cf. Eqs. (8.84) or (8.85)). A measurement of $\dot\nu_{\rm GW}$ at given $\nu_{\rm GW}$ then provides a measurement of the chirp mass according to Eq. (8.81). From the measurement of h_+ and/or h_+, one can then read off the luminosity distance d_L.

In more detail, defining the characteristic timescale for the GW frequency to change,

$$\tau_\nu \equiv \frac{\nu_{\rm GW}}{\dot\nu_{\rm GW}} \overset{(8.81)}{=} \frac{5}{96\pi^{8/3}}\left(\frac{c^3}{GM_c}\right)^{5/3}\nu_{\rm GW}^{-8/3}, \tag{8.95}$$

one can re-express the amplitudes of the waveforms as (cf. Eqs. (8.74) or (8.84))

$$\bar h_+(t) = \frac{1}{d_L}\frac{5c}{24\pi^2}\frac{1}{\tau_\nu \nu_{\rm GW}^2}\frac{1+\cos^2\iota}{2}, \tag{8.96}$$
$$\bar h_\times(t) = \frac{1}{d_L}\frac{5c}{24\pi^2}\frac{1}{\tau_\nu \nu_{\rm GW}^2}\cos\iota. \tag{8.97}$$

The key to this method is that the component masses m_1 and m_2 enter both the expressions for the frequency derivative $\dot\nu_{\rm GW}$ (cf. Eq. (8.81)) and the waveforms $\bar h_+$ and $\bar h_\times$ in the exact same way, namely through the chirp mass M_c. The fact that these waveforms can be expressed independently of the binary component masses (or chirp mass) makes these systems '*standard* *candles*' (often called '*standard sirens*' in this case[9])—the signal amplitudes are entirely determined by τ_ν at given $\nu_{\rm GW}$, the inclination angle, and the luminosity distance. In other words, the instantaneous intrinsic luminosity of the source is specified entirely by τ_ν at given $\nu_{\rm GW}$ without additional assumptions beyond the validity of general relativity. In fact, as pointed out by Forward & Berman (1967), one can easily show with the above formulae that the total power radiated by a binary system (i.e. its instantaneous luminosity) is only a function of the absolute phase Φ before collapse,

$$P_{\rm GW} = P_{\rm GW}(\Phi) = \frac{c^5}{160}\Phi^{-2}, \tag{8.98}$$

independent of the total mass and the mass ratio. This means that all binaries undergo the same history of power output as a function of radians before collapse. This result captures the theoretical foundation for standard sirens, and it is somewhat surprising that it took approximately twenty years until this application of binary systems was pointed out by Schutz (1986).

Exercise 8.4.1. *Derive Eq.* (8.98).

Given measurements of τ_ν, $\bar{h}_+(t)$, and $\bar{h}_\times(t)$, the ratio $\bar{h}_+(t)/\bar{h}_\times(t)$ determines the inclination $\cos\iota$, and one obtains the luminosity distance from either waveform:

$$d_L = \frac{5c}{24\pi^2}\frac{1}{\bar{h}_+(t)}\frac{1}{\tau_\nu \nu_{\rm GW}^2}\frac{1+\cos^2\iota}{2}, \tag{8.99}$$

$$d_L = \frac{5c}{24\pi^2}\frac{1}{\bar{h}_\times(t)}\frac{1}{\tau_\nu \nu_{\rm GW}^2}\cos\iota. \tag{8.100}$$

Note that no empirical calibration is required for this method—for example, no reference to the cosmic distance ladder or similar is required. The only empirical calibration required is that of the GW detector itself.

If the source also emits electromagnetic radiation as the binary merges, such as in the case of binary neutron stars, the host galaxy may be identified and one can additionally infer the redshift z of the source through spectroscopic analysis of the host galaxy spectrum (observation of line redshifts). Under certain circumstances, the redshift may also be directly inferred from an electromagnetic counterpart signal. Therefore, measuring (d_L, z) for sufficiently many gravitational-wave events allows one to measure the Hubble parameter $H(z)$ (cf. Eq. (8.94)). For low-redshift sources ($z \ll 1; H(z) \approx H_0$), Eq. (8.94) reduces to

$$z \simeq H_0 \frac{d_L}{c}, \tag{8.101}$$

and simultaneous measurements of redshift and luminosity distance results in a measurement of the Hubble constant H_0,

$$H_0 \simeq \frac{cz}{d_L}. \tag{8.102}$$

The first such standard siren measurement of the Hubble constant has been conducted with the first observed binary neutron-star merger event GW170817 (Abbott et al. 2017b).

[9]This is because sound is often used as an analogy when comparing GWs to electromagnetic radiation (light). Furthermore, this allows GW sources to be distinguished from Supernovae Type Ia, which are referred to as standard candles in the context of the cosmic distance ladder.

Remark (peculiar velocities). In practice, the velocity of a galaxy relative to the observer is the sum of a velocity due to cosmological expansion, $v_H = cz$ (the 'Hubble flow'), and its peculiar velocity v_p due to motion within a galaxy cluster, relative to the Hubble flow. Equation (8.102) thus becomes

$$H_0 \simeq \frac{cz + v_\mathrm{p}}{d_L}. \tag{8.103}$$

Since typical peculiar velocities are of the order of $\sim 300\,\mathrm{km\,s^{-1}}$, they can represent a $v_\mathrm{p}/(cz) = v_\mathrm{p}/(H_0 d_L) \sim \mathcal{O}(10\%)$ correction to the measurement of the Hubble constant for nearby sources at tens of Mpc. Here, we have assumed $H_0 \approx 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ and $d_L \simeq 40\,\mathrm{Mpc}$. In the case of GW170817, $v_H = (3327 \pm 72)\,\mathrm{km\,s^{-1}}$, $v_\mathrm{p} = (310 \pm 150)\,\mathrm{km\,s^{-1}}$, and $d_L \simeq 40\,\mathrm{Mpc}$ (Abbott et al. 2017b). Thus, the peculiar velocity of the host galaxy NGC 4993 is about 10% of its local Hubble flow and represents a crucial ingredient for measuring the Hubble constant.

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